HIGH SCHOOL MATHEMATICS

Unit 8.

SEQUENCES

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

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PREFACE

In Unit 7 you discovered basic principles $[(I^+_1) - (I^+_3)]$ for the positive integers and, on the basis of these, developed the procedures of definition by recursion and proof by induction. [You also discovered other basic principles $[(P_1) - (P_4)$ and $(G)]$ which enabled you to organize your knowledge of inequations.]

Using recursive definition and mathematical induction you were able to prove a variety of theorems about positive integers--closure theorems, the least number theorem, theorems concerning the greatest integer function and the divisibility relation, and theorems about polygonal numbers and about the number, $C(n, p)$, of p-membered subsets of an n-membered set. Among the theorems you proved were six of the seven assumptions [see page 7-47] which you made about positive integers in Unit 4. The seventh of these assumptions--that each positive integer other than 1 has just one prime factorization--will be proved in this unit.

The Introduction to Unit 7 proposed seven problems each of which depended, in some way or another, on properties of positive integers. Only one of them [Problem VII] was solved in that unit. In the present unit you will use what you learned in Unit 7 to develop concepts and techniques which will make it easy to solve the other six problems and problems like them. You will also learn a good deal more about <u>combinatorial</u> problems like Problem VII.

Most of the problems in the Introduction have to do with adding terms of a sequence of numbers, and about half of this unit deals with such "continued sums". The other half deals with "continued products" and, in particular, with powers and exponents. Among other things, you will learn how to justify the techniques you learned in Unit 4 for simplifying exponential expressions, and will learn more techniques for factoring.

As in Unit 7, you should take full advantage of the opportunities offered by the Miscellaneous Exercises and the Review Exercises for practicing old and new techniques.

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Sequences.--"What is the sum of the first 60 positive odd numbers?" This sounds like an impossibly tedious problem, but there is an easy way to attack it. Consider the sequence of positive odd numbers

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \quad 15, \quad 17, \quad \ldots .$$

The sum of the first 2 of these is 4, the sum of the first 3 is 9, the sum of the first 4 is 16, etc. Let's examine these successive sums and see if we can find a pattern.

$$\begin{array}{ll} 1 & = 1 \\ 1+3 & = 4 \\ 1+3+5 & = 9 \\ 1+3+5+7 & = 16 \\ 1+3+5+7+9 & = 25 \end{array}$$

Can you find the next sum without actually adding?

$$1+3+5+7+9+11 =$$

Continue finding sums until you can tell the sum of the first 60 odd positive numbers without hesitation. Then tell the sum of the first 90 odd positive numbers.

Here is another sequence of numbers.

$$\frac{1}{1\cdot 2}, \quad \frac{1}{2\cdot 3}, \quad \frac{1}{3\cdot 4}, \quad \frac{1}{4\cdot 5}, \quad \cdots$$

How many terms of this sequence starting with the first must you add to get a sum greater than 1? Begin by considering sums of the first few terms and try to find a pattern.

$$\begin{array}{ll} \frac{1}{1\cdot 2} & = \\[2ex] \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3} & = \\[2ex] \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4} & = \\[2ex] \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5} & = \end{array}$$

What do you think is the sum of the first 5 terms? The first 6 terms? The first 1000 terms? The first 1000000 terms? Do you think that there is such a sum greater than 1?

In Unit 7 you studied functions whose domain is I^+, the set of positive integers. Such functions are called <u>sequences</u>. The values of such a function are called its <u>terms</u>. When we speak of the <u>first</u> term of a sequence, we mean the value of the function for the argument 1. Similarly, the fourth term of the sequence is the value for the argument 4. In general, for each n, the nth term of the sequence is the value for the argument n.

For example, consider the sequence O of positive odd numbers.

$$O = \{(x, y), x \in I^+ : y = 2x - 1\}$$

Its fourth term, O_4, is 7. What is O_{96}? [What is O_{-3}?]

When one can find a formula for the nth term of a sequence, the formula can be used to give an <u>explicit</u> definition. For example, O is the sequence such that

$$\forall_n \; O_n = 2n - 1.$$

The sequence O may also be defined <u>recursively</u>:

$$\begin{cases} O_1 = 1 \\ \forall_n \; O_{n+1} = O_n + 2 \end{cases}$$

We shall use 'a' and 'b' as variables whose values are sequences. So, for example, we can say that

O is the sequence a such that, for each n, $a_n = 2n - 1$.

Also, returning to the sequence in the second example on page 8-1, we can say that this is the sequence a such that

$$\forall_n \; a_n = \frac{1}{n(n + 1)}.$$

Give a recursive definition for this sequence.

EXERCISES

1. What are the 8th, 9th, and 10th terms of the sequence a such that, for each n, $a_n = n(n - 1)(n - 2)$?

2. If, for each n, $b_n = \frac{n - 1}{2n + 1}$ then, for each n, $b_{n+1} =$, and $b_{n+2} =$

3. For each m, the mth positive even number is and the 3mth positive odd number is

4. If, for each p, $b_p = p^2 - 3p + 5$, what term of the sequence b is 75?

5. If $s_1 = 1$ and, for each q, $s_{q+1} = s_q + (2q + 1)$, compute s_{60}.

6. What is the sixtieth term of the sequence a such that $a_1 = \frac{1}{2}$ and, for each n, $a_{n+1} = a_n \cdot \frac{n+1}{n+2}$?

7. If the nth term of a sequence a is $(n + 1)(n + 2)(n + 3)$, what is the $(n + 1)$th term? The $(n + 2)$th term?

8. Give a recursive definition for the sequence a such that for each $r \in I^+$, $a_r = 5r - 1$.

9. If the nth term of a sequence is $3n - 4$, what is the difference of the 2nth term from the $(2n + 1)$th term?

10. Consider the sequence b such that, for each m, $b_m = 1$. What is the sum of the first 10 terms of b?

11. What is the largest term of the sequence a such that, for each n, $a_n = 3 - 10n$? What is the smallest term?

12. What are the smallest and largest terms of the sequence a where, for each m, $a_m = 3 - (m - 5)^2$? How many terms of a are positive?

☆13. Suppose that, for each n, s_n is the sum of the first n terms of the sequence a where, for each n, $a_n = 4n^2 - 36n + 71$. Find an integer m such that $s_{2m} = 2s_m$.

14. Suppose that b is a sequence such that, for each n, $b_{2n-1} = 1$ and $b_{2n} = 2$. What is the sum of the first 1000 terms of b?

15. Let a be a sequence such that, for each n, $a_n = (-1)^{n+1}$. What is the sum of the first 100 terms of a?

16. What is the sum of the first 100 positive odd integers? Of the second 100 positive odd integers?

8.01 <u>Continued</u> <u>sums</u>. --You probably had no trouble in doing the first problem on page 8-1 on finding the sum of the first 60 positive odd numbers. In fact, you probably guessed an interesting generalization:

(*) $\forall_n$ the sum of the first n positive odd integers is n^2

Problems which deal with sums of consecutive terms of a sequence occur frequently in mathematics. In order to deal with them conveniently, it is customary to use a special notation.

SIGMA-NOTATION

Instead of (*) we might have written:

$\forall_n$ the sum of the numbers O_p, for all $p \leq n$, is n^2,

and, recalling the explicit definition of O, this can be written:

$\forall_n$ the sum of the numbers $2p - 1$, for all $p \leq n$, is n^2

It is customary, instead of:

(1) $\underbrace{\text{the sum of the numbers}}\ \underbrace{2p - 1,}\ \underbrace{\text{for all positive integers } p \leq n}$

to write:

(2) $$\sum_{p=1}^{n} (2p - 1)$$

The sign 'Σ' is the Greek capital letter sigma, and is meant to suggest the first letter of the word 'sum'. The expression (2) can be read as (1), or, more expeditiously, as:

sigma, from $p = 1$ to n, of $2p - 1$

Using this notation, the conclusion about sums of odd numbers is:

$$\forall_n \sum_{p=1}^{n} (2p - 1) = n^2$$

This generalization we have discovered has for one of its instances:

$$\sum_{p=1}^{5} (2p - 1) = 5^2$$

The left side of this equation can be expanded to:

$$(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) + (2 \cdot 5 - 1)$$

and this simplifies to:

$$1 + 3 + 5 + 7 + 9$$

So, the instance is a short way of saying that $1 + 3 + 5 + 7 + 9 = 25$.

EXERCISES

A. Rewrite each of the following in sigma-notation.

1. For each n, the sum of the numbers $2p + 1$, for all $p \leq n$, is $n^2 + 2n$.

2. The sum of the numbers $\frac{1}{7q + 3}$, for all $q \leq 2$, is $\frac{27}{170}$.

3. The sum, from $m = 1$ to 5, of $\frac{1}{m^2 + 1}$ is not 81.

4. Sigma, from $q = 1$ to 2, of $\frac{1}{q}$ is 1.5.

5. The sum of the numbers $\frac{1}{p^2}$, for all $p \leq 1$, is 1.

B. Rewrite each of the following in the form of Exercise 4 of Part A.

1. $\sum_{p=1}^{7} (3p - 4) \geq 0$

2. $\sum_{p=1}^{8} p \neq \sum_{p=1}^{9} p$

C. Expand each of the following.

1. $\sum_{m=1}^{5} m$

2. $\sum_{q=1}^{3} (2 - 3q)$

3. $\sum_{n=1}^{3} (2 - 3n)$

4. $\sum_{p=1}^{4} \frac{p(p-1)}{2}$

5. $\sum_{m=1}^{5} \frac{1}{m(m+1)}$

6. $\sum_{q=1}^{5} \left(\frac{1}{q} - \frac{1}{q+1}\right)$

7. $\sum_{n=1}^{10} 4n$ 8. $\sum_{p=1}^{4} (3p + 4)$ 9. $\sum_{p=1}^{6} (0 \cdot p + 4)$

D. Verify each of the following.

1. $\sum_{p=1}^{8} p = \frac{8 \cdot 9}{2}$ 2. $\sum_{q=1}^{10} \left(\frac{1}{q} - \frac{1}{q+1}\right) = \frac{10}{11}$ 3. $\sum_{p=1}^{6} 4 = 24$

E. Express each of the following in Σ-notation.

Sample 1. $(2 + 3 \cdot 1) + (2 + 3 \cdot 2) + (2 + 3 \cdot 3) + (2 + 3 \cdot 4)$

Solution. $\sum_{p=1}^{4} (2 + 3p)$

1. $(5 \cdot 1 + 2) + (5 \cdot 2 + 2) + (5 \cdot 3 + 2) + (5 \cdot 4 + 2)$

2. $(1 - 3 \cdot 1^2) + (1 - 3 \cdot 2^2) + (1 - 3 \cdot 3^2) + (1 - 3 \cdot 4^2) + (1 - 3 \cdot 5^2)$

3. $2 + 4 + 6 + 8 + 10 + 12$ 4. $1 + 8 + 27 + 64 + 125$

5. $4 + 4 + 4 + 4 + 4 + 4$ 6. $(1 - 1) + (4 - 2) + (9 - 3)$

Sample 2. $\frac{-1}{2} + \frac{0}{3} + \frac{1}{4} + \frac{2}{5}$

Solution. $\sum_{p=1}^{4} \frac{p - 2}{p + 1}$

7. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5}$ 8. $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6}$

9. $\frac{1}{2 \cdot 3} + \frac{4}{4 \cdot 5} + \frac{9}{6 \cdot 7} + \frac{16}{8 \cdot 9} + \frac{25}{10 \cdot 11}$

10. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + 6 \cdot 7 \cdot 8$

11. $1 + 3 + 5 + \ldots + (2p - 1) + \ldots + 79$

12. $5 + 8 + 11 + \ldots + (3p + 2) + \ldots + 35$

13. $2 + 16 + 54 + \ldots + 2q^3 + \ldots + 2000$

14. $2 + 9 + 28 + 65 + 126 + 217$

15. $1 + 2 + 9 + 28 + 65 + 126 + 217$

16. $8 + 64 + 216 + 512 + 1000$

17. $\sum_{p=1}^{7} (3p + 1) + (3 \cdot 8 + 1)$

18. $\sum_{p=1}^{5} (p^2 + 1) + 37$

19. $\sum_{p=1}^{5} (4p + 1) + 54$

20. $\sum_{p=1}^{n} p(p + 1) + (n + 1)(n + 2)$

21. $\sum_{p=1}^{n+3} (p - 1)(p + 1) + (n + 3)(n + 5)$

F. Verify each of the following.

1. $\sum_{q=1}^{1} (6 + 7q) = 13$

2. $\sum_{n=1}^{1} \frac{1}{n(n + 1)} = \frac{1}{2}$

3. $\sum_{m=1}^{76} m^2 = \sum_{m=1}^{75} m^2 + 76^2$

4. $\sum_{p=1}^{4} (2p - 1) = \sum_{p=1}^{3} (2p - 1) + (2 \cdot 4 - 1)$

G. How many integers m satisfy each of the following sentences?

1. $\sum_{p=1}^{6} p < m \leq \sum_{p=1}^{7} p$

2. $\sum_{p=1}^{32} p < m \leq \sum_{p=1}^{33} p$

3. $\sum_{p=1}^{10} (5p + 3) < m \leq \sum_{p=1}^{11} (5p + 3)$

4. $\sum_{p=1}^{15} p^2 < m < \sum_{p=1}^{16} p^2$

5. $\sum_{p=1}^{17} -p < m \leq \sum_{p=1}^{18} -p$

6. $\sum_{p=1}^{18} -p < m \leq \sum_{p=1}^{17} -p$

7. $\sum_{p=1}^{92} a_p < m \leq \sum_{p=1}^{93} a_p$, where a is a sequence of positive integers

SUMMING A SEQUENCE

In the opening section of this unit we raised a question concerning the continued sums of the sequence b such that, for each p, $b_p = \frac{1}{p(p+1)}$. These continued sums are the terms of a second sequence s.

$$s_1 = \sum_{p=1}^{1} b_p = \frac{1}{1 \cdot 2} \qquad = \frac{1}{2}$$

$$s_2 = \sum_{p=1}^{2} b_p = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} \qquad = \frac{2}{3}$$

$$s_3 = \sum_{p=1}^{3} b_p = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} \qquad = \frac{3}{4}$$

$$s_4 = \sum_{p=1}^{4} b_p = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} \qquad = \frac{4}{5}$$

We can define s recursively as follows:

$$\begin{cases} s_1 = \frac{1}{1 \cdot 2} \\ \forall_n \quad s_{n+1} = s_n + \frac{1}{(n+1)(n+2)} \end{cases}$$

[Can you guess an explicit definition for s?]

In general, the continued sums of a sequence are the terms of a second sequence. This second sequence is easily defined recursively

in terms of the first sequence. Suppose that a is any sequence. Then, the sequence of its continued sums is defined as follows:

$$\begin{cases} \sum_{p=1}^{1} a_p = a_1 \\ \forall_n \sum_{p=1}^{n+1} a_p = \sum_{p=1}^{n} a_p + a_{n+1} \end{cases}$$

Of course, finding a recursive definition for the sequence of continued sums of a given sequence is trivial. The harder job and the more interesting one is to discover and establish an explicit definition. An explicit definition makes it easy to compute any of the continued sums.

For example, consider the sequence b mentioned earlier.

$$\forall_n \; b_n = \frac{1}{n(n+1)}$$

Its continued sum sequence is defined recursively by:

$$\begin{cases} \sum_{p=1}^{1} \frac{1}{p(p+1)} = \frac{1}{1 \cdot 2} \\ \forall_n \sum_{p=1}^{n+1} \frac{1}{p(p+1)} = \sum_{p=1}^{n} \frac{1}{p(p+1)} + \frac{1}{(n+1)(n+2)} \end{cases}$$

If you wish to find the sum of the first 100 terms of b, the recursive definition <u>can</u> be used:

$$\sum_{p=1}^{99+1} \frac{1}{p(p+1)} = \sum_{p=1}^{99} \frac{1}{p(p+1)} + \frac{1}{100 \cdot 101}$$

$$= \sum_{p=1}^{98} \frac{1}{p(p+1)} + \frac{1}{99 \cdot 100} + \frac{1}{100 \cdot 101}$$

$$= \text{etc.!!}$$

But, as you have probably discovered, it is easier to experiment by computing the first few terms of the sum sequence and searching for a pattern. In this case, it appears to be the case that

$$\forall_n \sum_{p=1}^{n} \frac{1}{p(p+1)} = \frac{n}{n+1}.$$

So, if this is correct, the sum of the first 100 terms of b is 100/101.

In order to show that the generalization is correct, we use mathematical induction to derive it from the recursive definition of the sum sequence. As in all inductive proofs, we need to derive two things:

(i) $$\sum_{p=1}^{1} \frac{1}{p(p+1)} = \frac{1}{1+1}$$

(ii) $$\forall_n \left[\sum_{p=1}^{n} \frac{1}{p(p+1)} = \frac{n}{n+1} \Rightarrow \sum_{p=1}^{n+1} \frac{1}{p(p+1)} = \frac{n+1}{(n+1)+1} \right]$$

Part (i):

(1) $$\sum_{p=1}^{1} \frac{1}{p(p+1)} = \frac{1}{1 \cdot 2}$$ [recursive definition]

(2) $$\frac{1}{1+1} = \frac{1}{1 \cdot 2}$$ [theorem]

(3) $$\sum_{p=1}^{1} \frac{1}{p(p+1)} = \frac{1}{1+1}$$ [(1), (2)]

Part (ii):

(4) $$\sum_{p=1}^{q} \frac{1}{p(p+1)} = \frac{q}{q+1}$$ [inductive hypothesis]*

$$(5)\quad \forall_n \sum_{p=1}^{n+1} \frac{1}{p(p+1)} = \sum_{p=1}^{n} \frac{1}{p(p+1)} + \frac{1}{(n+1)[(n+1)+1]} \qquad \text{[recursive definition]}$$

$$(6)\quad \sum_{p=1}^{q+1} \frac{1}{p(p+1)} = \sum_{p=1}^{q} \frac{1}{p(p+1)} + \frac{1}{(q+1)[(q+1)+1]} \qquad [(5)]$$

$$(7)\quad = \frac{q}{q+1} + \frac{1}{(q+1)(q+2)} \qquad [(4),\ (6)]$$

$$(8)\quad \forall_n \frac{n+1}{n+2} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \qquad \text{[theorem]}$$

$$(9)\quad \frac{q+1}{q+2} = \frac{q}{q+1} + \frac{1}{(q+1)(q+2)} \qquad [(8)]$$

$$(10)\quad \sum_{p=1}^{q+1} \frac{1}{p(p+1)} = \frac{q+1}{q+2} \qquad [(7),\ (9)]$$

$$(11)\quad \sum_{p=1}^{q} \frac{1}{p(p+1)} = \frac{q}{q+1} \Rightarrow \sum_{p=1}^{q+1} \frac{1}{p(p+1)} = \frac{q+1}{q+2} \qquad [(10);\ {*}(4)]$$

$$(12)\quad \forall_n \left[\sum_{p=1}^{n} \frac{1}{p(p+1)} = \frac{n}{n+1} \Rightarrow \sum_{p=1}^{n+1} \frac{1}{p(p+1)} = \frac{n+1}{n+2} \right] \qquad [(4) - (11)]$$

Part (iii):

$$(13)\quad \forall_n \sum_{p=1}^{n} \frac{1}{p(p+1)} = \frac{n}{n+1} \qquad [(3), (12), \text{PMI}]$$

The only tricky thing about this proof is step (8). Try to prove:

$$\forall_n \frac{n+1}{n+2} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

In the proof of part (ii), we might have written:

(7) $$= \frac{q}{q+1} + \frac{1}{(q+1)(q+2)}$$ [(4), (6)]

(8) $$= \frac{q^2+2q+1}{(q+1)(q+2)}$$

(9) $$= \frac{q+1}{q+2}$$ [algebra]

(10) $$\sum_{p=1}^{q} \frac{1}{p(p+1)} = \frac{q}{q+1} \Longrightarrow \sum_{p=1}^{q+1} \frac{1}{p(p+1)} = \frac{q+1}{q+2}$$ [(9); *(4)]

etc.

The algebra steps actually provide a proof of the theorem in the original step (8). The original step (9) is the relevant instance of this theorem.

EXERCISES

A. Prove the conjecture made at the outset of this unit about the sums of consecutive positive odd numbers:

$$\forall_n \sum_{p=1}^{n} (2p - 1) = n^2$$

B. Use mathematical induction to prove each of the following.

1. $\forall_n \sum_{p=1}^{n} (5 + 4p) = n(2n + 7)$

2. $\forall_n \sum_{p=1}^{n} (3p - 2) = \frac{n(3n-1)}{2}$

3. $\forall_n \sum_{p=1}^{n} (2p - 1)^2 = \frac{n(2n-1)(2n+1)}{3}$

☆4. $\forall_n \sum_{p=1}^{n} \frac{1}{p^2} \leq 2 - \frac{1}{n}$

C. 1. Use Exercise 2 of Part B to find the sum of the first 100 terms of the sequence

$$1,\ 4,\ 7,\ 10,\ 13,\ 16,\ \ldots .$$

2. State and prove a theorem which would help you find the sum of the first 100 terms of the sequence

$$1,\ 5,\ 9,\ 13,\ 17,\ 21,\ \ldots .$$

3. Discover and prove a more general theorem of which these two are instances.

D. Let a be a sequence such that, for each p, $a_p = \frac{1}{(2p-1)(2p+1)}$. Prove:

$$\forall_n \sum_{p=1}^{n} a_p < \frac{1}{2}$$

[Hint. Compute some terms of the continued sum sequence, and search for a pattern. Then, state and prove a theorem of the form:

$$\forall_n \sum_{p=1}^{n} \frac{1}{(2p-1)(2p+1)} =$$

and use this to prove the desired result.]

☆E. Compare the theorem:

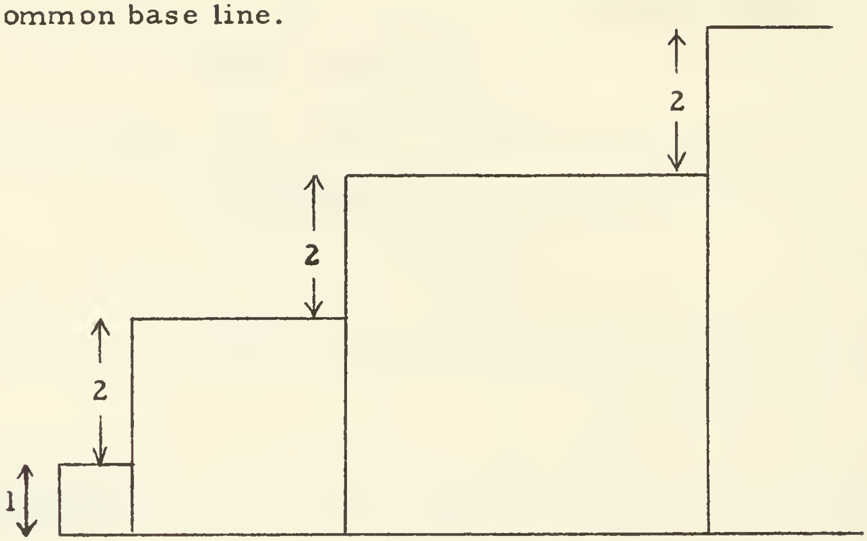
$$\forall_n \sum_{p=1}^{n} \frac{1}{p(p+1)} = \frac{n}{n+1}$$

proved in the text discussion with the theorem you discovered in Part D:

$$\forall_n \sum_{p=1}^{n} \frac{1}{(2p-1)(2p+1)} =$$

Discover and prove a more general theorem of which these two are instances.

F. A sequence of fifteen squares is constructed one next to the other along a common base line.



1. Find the sum of the area-measures of these fifteen squares. [Hint. One of the exercises in Part B on page 8-12 will help.]

2. Find the sum of their perimeters.

G. Consider the sequence a such that, for each p, $a_p = \frac{1}{p(p+1)(p+2)}$. Which of the sequences defined below are not the continued sum sequence for a?

(a) $\forall_p\ b_p = \frac{2p^2 + 3p + 1}{6(p+1)(p+2)}$ (b) $\forall_p\ b_p = \frac{6 + 19p - p^2}{48(p+2)}$

(c) $\forall_p\ b_p = \frac{p(p+3)}{4(p+1)(p+2)}$ (d) $\forall_p\ b_p = \frac{p(p+2)}{4(p+1)(p+3)}$

(e) $\begin{cases} b_1 = \frac{1}{6} \\ \forall_p\ b_{p+1} = b_p + \frac{1}{p(p+1)(p+2)} \end{cases}$ (f) $\begin{cases} b_1 = \frac{1}{6} \\ \forall_p\ b_{p+1} = b_p + \frac{1}{(p+1)(p+2)(p+3)} \end{cases}$

A SHORT CUT

You may have noticed that solutions of summation problems [like Exercises 1, 2, and 3 in Part B on page 8-12] are pretty much the same.

Let's see where the sameness is, and try to find a short cut. To do so, let's reexamine the proof on pages 8-10 and 8-11 of the summation theorem:

$$(*) \qquad \forall_n \sum_{p=1}^{n} \frac{1}{p(p+1)} = \frac{n}{n+1}$$

This theorem is of the form:

$$\forall_n \sum_{p=1}^{n} a_p = b_n$$

In (*) the sequences a and b are defined by:

$$\forall_p\, a_p = \frac{1}{p(p+1)} \qquad \text{and:} \qquad \forall_p\, b_p = \frac{p}{p+1}$$

Looking at the inductive proof, we see that in part (i) we used a theorem [step (2)]:

$$\frac{1}{1+1} = \frac{1}{1 \cdot 2},$$

and in part (ii) we used another theorem [step (8)]:

$$\forall_n \frac{n+1}{n+2} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

From the definitions of the sequences a and b, we see that these theorems amount to a recursive definition:

$$(\ddagger) \qquad \begin{cases} b_1 = a_1 \\ \forall_n\, b_{n+1} = b_n + a_{n+1} \end{cases}$$

of the sequence b in terms of the sequence a.

In steps (1) and (5) of the proof we used the recursive definition:

$$\begin{cases} \displaystyle\sum_{p=1}^{1} \frac{1}{p(p+1)} = \frac{1}{1 \cdot 2} \\[2ex] \displaystyle\forall_n \sum_{p=1}^{n+1} \frac{1}{p(p+1)} = \sum_{p=1}^{n} \frac{1}{p(p+1)} + \frac{1}{(n+1)(n+2)} \end{cases}$$

of the summation sequence for the sequence a--that is, we used:

$$\begin{cases} \sum_{p=1}^{1} a_p = a_1 \\ \forall_n \sum_{p=1}^{n+1} a_p = \sum_{p=1}^{n} a_p + a_{n+1} \end{cases}$$

Compare this recursive definition of the summation sequence for a with the recursive definition (⚓) of the sequence b. Do you see that the summation sequence and the sequence b are recursively defined in the same way in terms of the sequence a? So, our proof of (*) amounted to showing that the summation sequence and b are the same sequence because they satisfy the same recursive definition.

Let's rewrite the proof so that we can see this clearly.

<u>Part (i)</u>:

(1) $\sum_{p=1}^{1} a_p = a_1$ [recursive definition of summation sequence]

(2) $b_1 = a_1$ [theorem]

(3) $\sum_{p=1}^{1} a_p = b_1$ [(1), (2)]

<u>Part (ii)</u>:

(4) $\sum_{p=1}^{q} a_p = b_q$ [inductive hypothesis]*

(5) $\forall_n \sum_{p=1}^{n+1} a_p = \sum_{p=1}^{n} a_p + a_{n+1}$ [recursive definition of summation sequence]

(6) $\sum_{p=1}^{q+1} a_p = \sum_{p=1}^{q} a_p + a_{q+1}$ [(5)]

(7) $$= b_q + a_{q+1}$$ [(4), (6)]

(8) $$\forall_n\, b_{n+1} = b_n + a_{n+1}$$ [theorem]

(9) $$b_{q+1} = b_q + a_{q+1}$$ [(8)]

(10) $$\sum_{p=1}^{q+1} a_p = b_{q+1}$$ [(7), (9)]

(11) $$\sum_{p=1}^{q} a_p = b_q \Rightarrow \sum_{p=1}^{q+1} a_p = b_{q+1}$$ [(10); *(4)]

(12) $$\forall_n \left[\sum_{p=1}^{n} a_p = b_n \Rightarrow \sum_{p=1}^{n+1} a_p = b_{n+1} \right]$$ [(4) - (11)]

<u>Part (iii)</u>:

(13) $$\forall_n \sum_{p=1}^{n} a_p = b_n$$ [(3), (12), PMI]

Now, if we forget that 'a' and 'b' were used as names for the particular sequences referred to in (*), we can label steps (2) and (8) with '[assumption]' instead of '[theorem]'. Then, by discharging these assumptions, we have a proof of:

<u>Theorem 130.</u>

For any sequences a and b,

$$\left(b_1 = a_1 \text{ and } \forall_n\, b_{n+1} = b_n + a_{n+1}\right) \Rightarrow \forall_n \sum_{p=1}^{n} a_p = b_n$$

Let's apply this theorem to find an explicit definition of the summation sequence for the sequence of positive integers $[\forall_p\ a_p = p]$:

$$\forall_n \sum_{p=1}^{n} p = ?$$

In Unit 7 you studied the sequence T of triangular numbers. It was defined recursively by:

$$\begin{cases} T_1 = 1 & = a_1 \\ \forall_n \quad T_{n+1} = T_n + (n+1) = T_n + a_{n+1} \end{cases}$$

where a is the sequence of positive integers. So, Theorem 130 tells us that

$$\forall_n \sum_{p=1}^{n} p = T_n .$$

Since, as you proved in Unit 7,

$$\forall_n \quad T_n = \frac{n(n+1)}{2},$$

we see that

$$\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}.$$

As a second example, consider the problem of proving:

$$\forall_n \sum_{p=1}^{n} \frac{1}{p(p+1)(p+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

This will follow from Theorem 130 if we can prove:

(i) $\dfrac{1(1+3)}{4(1+1)(1+2)} = \dfrac{1}{1(1+1)(1+2)}$

and:

(ii) $\forall_n \dfrac{(n+1)(n+4)}{4(n+2)(n+3)} = \dfrac{n(n+3)}{4(n+1)(n+2)} + \dfrac{1}{(n+1)(n+2)(n+3)}$

We leave the proof of these algebra theorems to you. They involve only routine computing and manipulating.

EXERCISES

A. Practice using Theorem 130 in proving the following summation theorems.

1. $\forall_n \sum_{p=1}^{n} (3p + 5) = \frac{n(3n + 13)}{2}$

2. $\forall_n \sum_{p=1}^{n} (6p^2 + 12p) = n(n + 1)(2n + 7)$

3. $\forall_n \sum_{p=1}^{n} \frac{p^2}{(2p - 1)(2p + 1)} = \frac{n(n + 1)}{2(2n + 1)}$

B. 1. Complete the following table.

n	1	2	3	4	5
$\sum_{p=1}^{n} p^2$					
$\sum_{p=1}^{n} p$					
$\sum_{p=1}^{n} p^3$					

2. (a) We have already proved a summation theorem for the sequence of positive integers:

$$\forall_n \sum_{p=1}^{n} p = \frac{n(n + 1)}{2}$$

Use the table and this theorem to guess summation theorems for the sequences of squares and cubes. [Turn page for hint.]

$$\forall_n \sum_{p=1}^{n} p^2 = \frac{\qquad}{6} \qquad\qquad \forall_n \sum_{p=1}^{n} p^3 = \underline{\qquad\qquad}$$

[Hint. Compute quotients of corresponding entries in first two rows of your table.]

(b) Prove the two theorems you discovered in part (a).

3. Compute.

(a) $1 + 2 + 3 + \ldots + 10$

(b) $1^2 + 2^2 + 3^2 + \ldots + 1000^2$

(c) $\sum_{p=1}^{75} p$

(d) $\sum_{p=1}^{75} p - \sum_{p=1}^{60} p$

(e) $\sum_{p=1}^{48} p - \sum_{q=1}^{39} q$

(f) $\sum_{p=1}^{200} p - \sum_{p=1}^{100} p$

(g) $100^2 + 101^2 + 102^2 + \ldots + 1000^2$

(h) $100^3 + 101^3 + 102^3 + \ldots + 1000^3$

(i) $(1^2 + 1^3) + (2^2 + 2^3) + (3^2 + 3^3) + \ldots + (1000^2 + 1000^3)$

(j) $1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \ldots + 1000^2 \cdot 1001$

☆(k) $1 \cdot 2 + 2 \cdot 5 + 3 \cdot 10 + 4 \cdot 17 + 5 \cdot 26 + \ldots + 738$

4. Solve.

(a) $\sum_{p=1}^{976} p = n$

(b) $\sum_{p=1}^{n} p = 476776$

(c) $\sum_{p=1}^{n} p = 325$

(d) $\sum_{p=1}^{n} p = 544$

(e) $\sum_{p=1}^{n+1} p = 2211$

(f) $\sum_{p=1}^{n} p < 190$

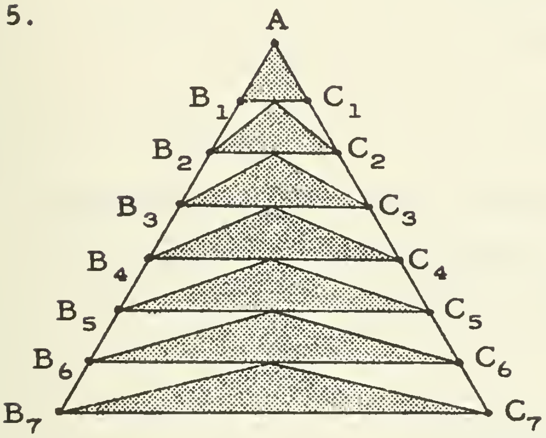
(g) $\sum_{p=1}^{n} p < 721$

(h) $\sum_{p=1}^{n-1} p < 721 < \sum_{p=1}^{n} p$

(i) $\sum_{p=1}^{n} p^3 = 90000$

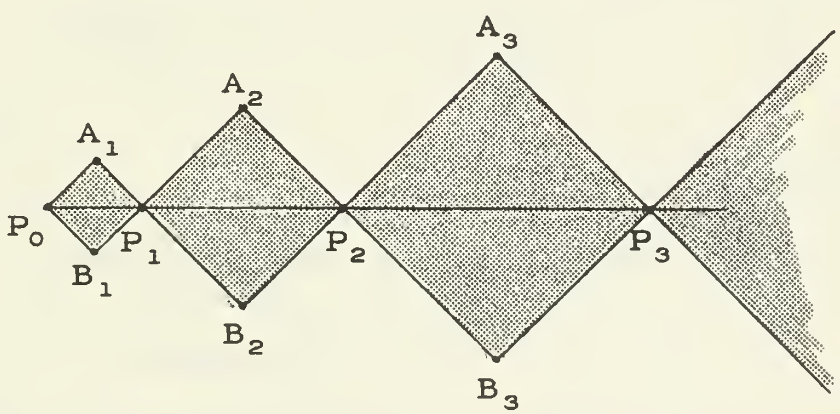
☆(j) $\sum_{p=1}^{n-1} p^3 < 3600 < \sum_{p=1}^{n} p^3$

5.



The shaded regions are bounded by isosceles triangles. $\angle B_1AC_1$ is an angle of 60°. Also, $AB_1 = B_1B_2 = B_2B_3 = \ldots = B_6B_7$, and $AC_1 = C_1C_2 = C_2C_3 = \ldots = C_6C_7$. If the area-measure of $\blacktriangle AB_1C_1$ is 1, what is the sum of the area-measures of the seven shaded regions?

6. Consider a sequence of ten squares drawn along a line in the manner shown. $P_0P_1 = 1$, $P_1P_2 = 2$, $P_2P_3 = 3$, ..., and $P_9P_{10} = 10$.



(a) Compute the sum of the area-measures of the ten squares.

(b) Compute the measure of the path $P_0A_1P_1B_2P_2A_3 \ldots P_1B_1P_0$.

(c) Are the points A_1, A_2, A_3, ..., and A_{10} collinear?

☆(d) Consider the coordinate system for which P_0 is the origin and P_1 is the unit-point. Find an equation whose graph contains the points A_1, A_2, ..., A_{10}, P_0, B_1, B_2, ..., and B_{10}.

7. Suppose that you continued writing terms in the sequence of fractions whose first seventeen terms are shown below.

$$\frac{1}{1}\ \frac{1}{2}\ \frac{2}{1}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}\ \frac{1}{4}\ \frac{2}{3}\ \frac{3}{2}\ \frac{4}{1}\ \frac{1}{5}\ \frac{2}{4}\ \frac{3}{3}\ \frac{4}{2}\ \frac{5}{1}\ \frac{1}{6}\ \frac{2}{5}\ \cdots$$

Notice that the sixth term in this sequence is '$\frac{3}{1}$' and that the nineteenth term is '$\frac{4}{3}$'. Study these terms until you see the pattern.

(a) Which term in the sequence is '$\frac{823}{471}$'?

☆(b) Which fraction is the 743rd term?

* * *

Sometimes a proof of one theorem will suggest a proof of another theorem. Recall that this is how we found a proof for Theorem 130. We can see another example of this in the following.

If we had proved the theorem:

$$\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}$$

by induction, or by a direct use of Theorem 130, we would have used the algebra theorem:

$$(*) \qquad \forall_n \ \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

[In fact, we did use this theorem in Unit 7 when we proved: $\forall_n\ T_n = \frac{n(n+1)}{2}$] Theorem (*) is very easy to prove:

$$\frac{q(q+1)}{2} + (q+1) = \frac{q(q+1) + 2(q+1)}{2}$$

$$= \frac{(q+1)(q+2)}{2}$$

The proof just given suggests another algebra theorem similar to (*):

$$(**) \quad \forall_n \ \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}.$$

and this suggests another summation theorem. Now go on to Part C.

* * *

C. 1. (a) Prove (**).

(b) Use Theorem 130 and (**) to prove a summation theorem which begins with:

$$\forall_n \sum_{p=1}^{n} p(p+1) =$$

2. (a) Prove the "next" algebra theorem like (*) and (**).

(b) Use this to prove another summation theorem.

3. Compute.

(a) $1 + 2 + 3 + 4 + \ldots + 100$

(b) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + 100 \cdot 101$

(c) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + 100 \cdot 101 \cdot 102$

(d) $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + \ldots + 100 \cdot 101 \cdot 102 \cdot 103$

(e) $2 \cdot 4 \cdot 6 + 4 \cdot 6 \cdot 8 + 6 \cdot 8 \cdot 10 + \ldots + 200 \cdot 202 \cdot 204$

4. A boy earns 2 cents for the first minute of work, 3 cents for each of the next 2 minutes, 4 cents for each of the next 3 minutes, 5 cents for each of the next 4 minutes, etc. At this strange pay rate, how much would he earn for 2 hours of work? ☆For 3 hours of work?

D. In Part B you found that the summation theorem for the sequence of positive integers is one of a sequence of summation theorems $[\Sigma p, \Sigma p^2, \Sigma p^3, \ldots]$. In Part C you found that it is also one of another sequence of summation theorems $[\Sigma p, \Sigma p(p+1), \Sigma p(p+1)(p+2), \ldots]$. Actually, it is the second theorem of each of these sequences. The first theorem begins:

$$\forall_n \sum_{p=1}^{n} 1 =$$

[Compare with Exercise 5 of Part E on page 8-6.]

Complete this theorem and prove it.

* * *

The results referred to in Part D can be collected into two theorems:

Theorem 131.

a. $\forall_n \sum_{p=1}^{n} 1 = n$

b. $\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}$

c. $\forall_n \sum_{p=1}^{n} p^2 = \frac{n(n+1)(2n+1)}{6}$

d. $\forall_n \sum_{p=1}^{n} p^3 = \frac{n^2(n+1)^2}{4}$

Theorem 132.

a. $\forall_n \sum_{p=1}^{n} 1 = n$

b. $\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}$

c. $\forall_n \sum_{p=1}^{n} p(p+1) = \frac{n(n+1)(n+2)}{3}$

d. $\forall_n \sum_{p=1}^{n} p(p+1)(p+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

It is easy to guess succeeding parts of Theorem 132, but not easy for Theorem 131.

* * *

☆E. The Fibonacci sequence is defined by:

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ \forall_n \; f_{n+2} = f_n + f_{n+1} \end{cases}$$

1. Compute the first ten terms of the Fibonacci sequence.

2. Evaluate $\sum_{p=1}^{n} f_p$ for n = 1, 2, 3, etc., comparing your results with with those of Exercise 1. Continue until you can guess a theorem:

$$\forall_n \sum_{p=1}^{n} f_p =$$

3. Prove the theorem you discovered in Exercise 2.

4. Guess, and prove, a theorem which begins:

$$\forall_n \sum_{p=1}^{n} f_{2p-1} =$$

5. Guess, and prove, a theorem which begins:

$$\forall_n \sum_{p=1}^{n} f_p + \sum_{p=1}^{n+1} f_p =$$

6. Fibonacci was a thirteenth-century Italian mathematician. In one of his works he proposed the following problem:

   Assume that a pair of adult rabbits can produce a new-born pair each month, and that it takes one month for a new-born pair to become adult. How many pairs of rabbits can be produced from an adult pair in one year?

   Solve this problem.

   [Discussions of the Fibonacci sequence and of some of its applications are in Edna E. Kramer's The Main Stream of Mathematics (New York: Oxford University Press, 1951) and in James R. Newman's The World of Mathematics, Volume 1 (New York: Simon and Schuster, 1956).]

* * *

MISCELLANEOUS EXERCISES

A. In each of the following exercises, you are given two sequences. Determine which corresponding terms, if any, are equal.

Sample. a: 7, 10, 13, ..., $3p + 4$, ...
b: 11, 13, 15, ..., $2p + 9$, ...

Solution. It is easy to see that the fifth term of sequence a is 19, and that the fifth term of sequence b is 19, also. So, $a_5 = b_5$.

Do you see a quick way of finding out which terms are equal? Can you prove that no other corresponding terms are equal?

1. a: 8, 14, 20, ..., $6p + 2$, ...
   b: −10, −2, 6, ..., $8p - 18$, ...

2. a: 9, 13, 17, ..., $4p + 5$, ...
   b: 6, 13, 20, ..., $7p - 1$, ...

3. a: 9, 14, 19, ..., $5p + 4$, ...
   b: 10, 13, 16, ..., $3p + 7$, ...

4. a: −6, −4, −2, ..., $2p - 8$, ...
   b: 6, 11, 16, ..., $5p + 1$, ...

5. $a_p = 9p + 6$, $b_p = 10p - 1$

6. $a_p = 9$, $b_p = 2p + 1$

7. $a_p = 3(p + 5) - 2$, $b_p = 7p - 3$

8. $a_p = 9(2p - 1) + 4$, $b_p = 3(4p - 2) + 6p + 1$

9. a: 73, 72, 77, ..., $3p^2 - 10p + 80$, ...
   b: 7, 22, 41, ..., $2p^2 + 9p - 4$, ...

10. $a_p = 3p^2 - p + 10$, $b_p = 2p^2 + 4p + 5$

11. $a_p = p^2 + 2p - 2$, $b_p = 2p^2 + 3p - 8$

12. $a_p = 2p^2 + p + 5, \qquad b_p = p^2 - 3p + 2$

13. $a_p = \frac{(p - 3)(p + 7)}{(p + 2)(p + 1)}, \qquad b_p = \frac{p + 2}{p + 4}$

B. Prove.

1. $\forall_n \sum_{p=1}^{n} (p + 1)(p + 4) = \frac{n(n + 4)(n + 5)}{3}$

2. $\sum_{p=1}^{r} (p^3 - p) = \frac{r(r^2 - 1)(r + 2)}{4}$, for all integral $r > 0$

3. $\forall_n \sum_{p=1}^{n} (p^2 - 1) = \frac{n(n - 1)(2n + 5)}{6}$

4. $\forall_n \sum_{p=1}^{n} p(p + 2) = \frac{n(n + 1)(2n + 7)}{6}$

5. $\forall_n \sum_{p=1}^{n} p(p + 1)(2p + 1) = \frac{n(n + 1)^2(n + 2)}{2}$

6. $\forall_n \sum_{p=1}^{n} (3p + 1)^2 = \frac{n(6n^2 + 15n + 11)}{2}$

7. $\forall_k \forall_n \sum_{p=1}^{n} p(p + k) = \frac{n(n + 1)(2n + 1 + 3k)}{6}$

8. $\forall_k \forall_n \sum_{p=1}^{n} p(p + k)(p + 2k) = \frac{n(n + 1)(n + 2k)(n + 2k + 1)}{4}$

9. $\forall_k \forall_n \sum_{p=1}^{n} p(p + k)(2p + k) = \frac{n(n + 1)(n + k)(n + k + 1)}{2}$

C. 1. Mr. Derber invests a total of \$1000, part at 5% and the rest at $3\frac{1}{2}$%. The income received from the 5% investment is just \$1 less per year than the income from the other investment. How much does Mr. Derber invest in each enterprise?

2. Simplify.

(a) $\frac{5j}{j^2 + 3j + 2} - \frac{j}{j^2 - 1}$ (b) $\frac{q^2}{q^2 + 3q + 4} - \frac{q + 7}{q + 2}$

3. Solve these equations.

(a) $\frac{1}{x + 1} - \frac{1}{x - 1} = 1$ (b) $\frac{1}{x} + \frac{1}{x + 2} = \frac{2}{x - 1}$

4. In a parallelogram, one of the acute angles is a complement of the other. How many degrees are there in each of the obtuse angles?

5. Factor.

(a) $(y - 1)^2 - x^2$ (b) $(t + u)^2 - s^2$ (c) $(k - 1)^4 - 1$

(d) $x^2 - 11x + 24$ (e) $3x^2 - 27$ (f) $\frac{1}{4}x^3 - \frac{1}{4}x$

(g) $4t^2 + 16t + 16$ (h) $3a^2 + 84a + 588$ (i) $6 - y - y^2$

(j) $a^2 - 17ab + 60b^2$ (k) $x^2 + xy - 90y^2$ (ℓ) $2x^2 + x - 6$

6. The sum of the measures of two adjacent sides of a rectangle is 16. If its area-measure is 48, what is its perimeter?

7. Solve these equations.

(a) $\frac{x - 5}{x - 4} = \frac{3x}{3x + 1}$ (b) $\frac{4}{x + 1} - \frac{x}{3 - x} = \frac{x^2 - 2x + 2}{x^2 - 2x - 3}$

8. Complete: $\forall_{x \neq 1}\ \frac{2x^2 - 3x - 5}{(x^2 + 2)(x - 1)} = \frac{(\quad)x + \underline{\quad}}{x^2 + 2} + \frac{\underline{\quad}}{x - 1}$

9. Simplify.

(a) $\frac{1}{a - b} + \frac{b}{a^2 - ab} - \frac{a}{(a - b)^2}$ (b) $3t + \frac{5t^2}{t - u} + t - u$

10. A diameter of circle C_1 is 20 inches longer than a diameter of circle C_2. If a radius of C_1 is r inches long, find the ratio of the circumference of C_1 to that of C_2.

11. Suppose that f is a function defined by: $f(x) = \frac{(3 - x)(x + 4)}{(2x - 1)(x + 1)}$

(a) For what real numbers is f undefined?

(b) Is f(2) positive? (c) Is f(−3) positive?

(d) Is f(−83) positive? (e) Is f(3) positive?

(f) Solve: $f(x) > 0$

12. Suppose that $g_n = n^2 + 4n - 21$. For what arguments of g, if any, does g have negative values?

13. Complete: $\forall_x \; 3x^3 - 10x^2 - x + 1 = 3x^3 + (\qquad\qquad)$

14. Prove that a triangle whose side-measures are 50, 50, and 60 has the same area-measure as a triangle whose side-measures are 50, 50, and 80.

15. Solve these systems of equations.

(a) $\begin{cases} \frac{2a}{5} - \frac{3b}{2} = 5 \\ \frac{a}{5} + \frac{3b}{2} = -2 \end{cases}$ (b) $\begin{cases} \frac{9 - 7x}{2} = y \\ 10 - y = \frac{5}{3}x \end{cases}$

16. If the side-measure of a square is increased by 3, the area-measure will be increased 39. What is the perimeter of the original square?

17. Simplify.

(a) $2 \times \sqrt[3]{0.027}$ (b) $7\frac{1}{2} \div 1\frac{1}{5}$ (c) $1 \div [10 + 9 \div (8 + 7)]$

18. Solve these equations.

(a) $\frac{1}{3}\%(x) = 24$ (b) $\sqrt{7} + \sqrt{13} = x(\sqrt{28} + \sqrt{52})$ (c) $2^x = 4^3$

19. Is 5 a root of the equation '$8x(x + 5)(x^2 - 3)(10 + 2x) = 0$'?

20. How many bus tokens can be bought for c dollars if tokens are three for a half-dollar?

21. If the perimeter of a square is $10x$, what is the area-measure?

22. Simplify.

(a) $\frac{20x^3y^2}{-4xy^2}$ (b) $\frac{42x^2 + 70xy}{7x}$ (c) $\frac{6y^3 - 13y^2 + 12y - 4}{3y - 2}$

23. Simplify.

(a) $\frac{-56(m + n)^3}{8(m + n)^2}$ (b) $\frac{a^4 - b^4}{a + b}$ (c) $\frac{(-3a^3b^2)(5a^4b^2c)}{(2ab^3)(-7a^2bc)}$

24. How many ounces does a t-pound block weigh?

25. Two similar polygons have areas of 72 square inches and 288 square inches, respectively. If a side of the smaller polygon is 3 inches long, what is the length of the corresponding side of the larger polygon?

26. Simplify.

(a) $20(x - y) - 8(x + y) - 10xy$ (b) $12(a + b) - 10(a + b) + 3a^2b^2$

(c) $(s + t) - 5(s - t) + 11(s - t) - (s + t)$

(d) $8a^2b + 3ab^2 - 5 - 4a^2b - 9ab^2 + 6$

27. Simplify.

(a) $\frac{5a - 5b}{2a + 2b} \times \frac{3a + 3b}{10a - 10b}$ (b) $\frac{x + y}{x^2 - y^2} \cdot \left(\frac{x - y}{x + y}\right)^2$

28. The graph of each equation given below is the graph of a linear function. Find the slope and the intercept of each function.

(a) $y = 7x + 3$ (b) $y = \frac{2x - 5}{7}$ (c) $7x - 3y - 6 = 0$

(d) $3(x - 2y) = 7 - 5y$ (e) $\frac{3x}{5} - \frac{2y}{7} = 1$ (f) $\frac{3 - x}{2} + \frac{5 + 2y}{3} = 0$

29. If $g(x) = 3x^2 - \frac{3x - 4}{4}$ then $g(2 - x) =$ ______.

30. Simplify.

(a) $4\sqrt{5} - \sqrt{20}$ (b) $\sqrt{75} + \sqrt{12} - \sqrt{45}$ (c) $\sqrt{35} \cdot \sqrt{7/5}$

(d) $(\sqrt{5} - \sqrt{3})^2 + (\sqrt{5} + \sqrt{3})^2$ (e) $(3\sqrt{6} - 5\sqrt{2})(7\sqrt{6} + 2\sqrt{2})$

31. A farmer has a certain length of fencing. He decides to construct a given number of rectangular pens by erecting two east-west fences and the necessary number of north-south fences. How must he apportion the fencing between north-south and east-west fences to enclose the largest area?

32. Solve for 'x': $px^2 + qx + r = 0$

33. Find the quadratic function which contains $(1, -3)$, $(3, 5)$, and $(-1, 5)$.

34. Solve these equations.

(a) $\frac{1}{x + 5} = \frac{1}{x + 4}$ (b) $\frac{x}{x - 5} - \frac{3}{x - 3} = \frac{21 - 5x}{x^2 - 8x + 15}$

EXPLORATION EXERCISES

1. Prove, for each sequence a, that

$$\forall_x \forall_n \sum_{p=1}^{n} xa_p = x \sum_{p=1}^{n} a_p .$$

2. Use the theorem just proved, and an earlier theorem, to prove:

$$\forall_x \forall_n \sum_{p=1}^{n} x = x \cdot n$$

3. Compute.

(a) $1 + 2 + 3 + \ldots + 50$ (b) $7 + 14 + 21 + \ldots + 350$

(c) $\frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \ldots + \frac{50}{3}$ (d) $\frac{1}{50} + \frac{2}{50} + \frac{3}{50} + \ldots + 1$

(e) $4 + 4 + 4 + \ldots + 4$ [50 terms]

EXTENDING THE MEANING OF Σ-NOTATION

The theorem:

$$(*) \qquad \forall_x \forall_n \sum_{p=1}^{n} x = x \cdot n$$

is worth a little study. One thing we may notice is that, in view of the recursive definition of Σ-notation, one of the consequences of (*) is:

$$\forall_x\ x \cdot 1 = x \qquad \text{[Explain.]}$$

Another consequence is:

$$\forall_x\ x \cdot 2 = x + x$$

In general, (*) says that multiplying by a positive integer is "repeated addition". To find the product of a number by 4 you can add the number to itself, then add the number to this first sum, and, finally, add the number to this second sum. [This may be the way multiplication was explained to you the first time you heard about it. Notice that this reduction of multiplication to addition works only for positive integers--you can't very well use it to multiply a number by $\sqrt{2}$, or even by $\frac{3}{4}$.]

The idea expressed by theorem (*) is a useful one, and a simple tag such as 'multiplication by a positive integer is repeated addition' helps one to remember it. The tag is useful, even though it doesn't make literal sense in all cases. Multiplication by 3, say, is repeated addition. But multiplication by 2 is simple addition [of the number to itself]--there is no repetition. And multiplication by 1 is "sameing"--there is no addition at all. Extending the meaning of 'repeated addition' to cover these cases is worthwhile because it gives us a simple way of putting into words what (*) says.

The definition of Σ-notation:

$$(\dagger) \qquad \sum_{p=1}^{1} a_p = a_1, \qquad \forall_n \sum_{p=1}^{n+1} a_p = \sum_{p=1}^{n} a_p + a_{n+1}$$

involves a similar extension of meaning. The second sentence of the definition says that, for each n,

> the sum of the first n + 1 terms of a sequence a is
> the sum of [the sum of the first n terms of a]
> and the (n + 1)th term of a.

As you have seen, in order to get started, we need the first sentence, which says that

> the sum of the first one terms of a sequence a is
> the first term of a.

Literally, the phrase 'the sum of the first...terms' makes sense only when the blank is filled by a numeral for an integer greater than 1. But it is convenient to extend the meaning, as is done in the first sentence of (†), so that it makes sense when the blank is filled by a numeral for 1. It is also convenient to extend the meaning still further so that it makes sense to say 'the sum of the first zero terms'. We do this by rewriting the definition of Σ-notation as follows:

$$(\dagger\dagger)\quad \begin{cases} \displaystyle\sum_{p=1}^{0} a_p = 0 \\ \forall_{k \geq 0} \displaystyle\sum_{p=1}^{k+1} a_p = \sum_{p=1}^{k} a_p + a_{k+1} \end{cases}$$

[Why 'k' instead of 'n'?]

One consequence of this is to enable us to include one more fact in (*) by writing:

$$(**)\quad \forall_x \forall_{k \geq 0} \sum_{p=1}^{k} x = x \cdot k$$

[What theorem is included in (**) which was not included in (*)?]

Notice that the new definition of Σ-notation implies the original one. For, from the second sentence of (††) we can conclude that

$$\sum_{p=1}^{1} a_p = \sum_{p=1}^{0} a_p + a_1,$$

from which, using the first sentence of (††), we obtain the first sentence of (†). And since, for each n, $n \geq 0$, the second sentence of (††) implies the second sentence of (†). So, nothing is lost if we take (††) as the recursive definition of Σ-notation for sequences. The only change is that, now, expressions of the form '$\displaystyle\sum_{p=1}^{0} \ldots$' are numerals for 0.

EXERCISES

A. Compute.

1. $\sum_{p=1}^{0} (3p + 5)$ 2. $\sum_{q=1}^{0} (2q^2 + 1)$ 3. $\sum_{m=1}^{0} (5m - \frac{1}{m})$

4. $\sum_{p=1}^{2} (p - 1)$ 5. $\sum_{p=1}^{1} (p - 1)$ 6. $\sum_{p=1}^{0} (p - 1)$

7. $\sum_{p=1}^{7} 5$ 8. $\sum_{p=1}^{0} 15$ 9. $\sum_{p=1}^{15} 0$

B. 1. Suppose that g is a function such that, for each $k \geq 0$, $g_k = 3k+7$. Compute g_0, g_1, g_2, and g_7.

2. Suppose that a is a function such that, for each $k \geq -7$, $a_k = 2k+1$. Compute a_{-7}, a_{-4}, a_0, and a_{-1}.

3. What is the sum of the first five values of the function described in Exercise 2?

4. Guess a meaning for the symbol:

$$\sum_{k=-7}^{-3} (2k + 1)$$

5. Guess a meaning for:

$$\sum_{p=3}^{7} [2(p - 10) + 1]$$

C. Express in Σ-notation.

1. $[(7 \cdot 1 + 2) + (7 \cdot 2 + 2) + (7 \cdot 3 + 2) + (7 \cdot 4 + 2) + (7 \cdot 5 + 2)] - [(7 \cdot 1 + 2) + (7 \cdot 2 + 2)]$

2. $(1 + 8 + 27 + 64 + 125) - (1 + 8 + 27)$

3. $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 - 2 - 4 - 6 - 8 - 10$

4. $16 + 25 + 36 + 49 + 64 + 81 + 100 + 121$

* * *

So far, we have dealt with continued sums of values of functions whose domain is I^+, and, more particularly, with sums of initial strings of values of such functions. In our later work we shall want to consider, also, functions whose domain is, for example, the set of nonnegative integers, and sums of consecutive values beginning with some term other than the first. There are two ways to do this.

For an example, consider the function a, whose domain is $I^+ \cup \{0\}$, such that

$$\forall_{k \geq 0}\ a_k = 2k + 3.$$

The first several values of this function are

$$3,\quad 5,\quad 7,\quad 9,\quad 11,\quad 13,\quad 15,\quad 17,\quad 19,\quad 21,\quad 23.$$

Now suppose that we wish to write a Σ-expression for the sum of the first, say, five values. We can do this by writing:

$$\sum_{p=1}^{5} [2(p - 1) + 3] \qquad \text{[Check this.]}$$

As a second problem, suppose that we wish to write a Σ-expression for the sum of five consecutive values of the function, beginning with the fourth [which is a_3]. We can express this sum by writing:

$$\sum_{p=1}^{5} [2(p + 2) + 3] \qquad \text{[Check this, also.]}$$

So, it is possible to use Σ-notation, as we have already defined it, to write about any sum of consecutive terms of a function whose domain is either I^+ or $I^+ \cup \{0\}$. In fact, we can, in the same way, deal with functions whose domains are any of the sets $\{k\colon\ k \geq j\}$, where $j \in I$.

However, a second, and often more convenient, method of expressing such sums is to use Σ-notation in a more general way.

Instead of writing:

$$\sum_{p=1}^{5} [2(p - 1) + 3]$$

we can write:

$$\sum_{k=0}^{4} (2k + 3)$$

and instead of writing:

$$\sum_{p=1}^{5} [2(p + 2) + 3]$$

we can write:

$$\sum_{k=3}^{7} (2k + 3) \qquad \left[\text{or: } \sum_{p=3}^{7} (2p + 3)\right]$$

To do this, we need to extend the definition of Σ-notation so that

$$\sum_{k=0}^{4} (2k + 3) = (2 \cdot 0 + 3) + (2 \cdot 1 + 3) + (2 \cdot 2 + 3) + (2 \cdot 3 + 3) + (2 \cdot 4 + 3)$$

and

$$\sum_{k=3}^{7} (2k + 3) = (2 \cdot 3 + 3) + (2 \cdot 4 + 3) + (2 \cdot 5 + 3) + (2 \cdot 6 + 3) + (2 \cdot 7 + 3).$$

This is easy to do.

For each $j \in I$ and each function a whose domain includes $\{k\colon\ k \geq j\}$,

$$\begin{cases} \displaystyle\sum_{i=j}^{j-1} a_i = 0 \\ \forall_{k \geq j-1} \displaystyle\sum_{i=j}^{k+1} a_i = \sum_{i=j}^{k} a_i + a_{k+1} \end{cases}$$

Note that the case $j = 1$ of this definition is (††) on page 8-33.

EXERCISES

A. Expand each of the following.

1. $\sum_{i=0}^{5} (3i - 4)$ 2. $\sum_{i=-2}^{3} (3i + 2)$ 3. $\sum_{i=3}^{8} (3i - 13)$

4. $\sum_{p=2}^{6} (p^2 - 1)$ 5. $\sum_{k=15}^{17} (31 - 2k)$ 6. $\sum_{i=-1}^{5} 1$

7. $\sum_{k=10}^{11} (k - 10)$ 8. $\sum_{k=10}^{10} (k - 10)$ 9. $\sum_{k=10}^{9} (k - 10)$

B. Each of the following Σ-expressions represents the sum of consecutive values of a function a. For each, tell how many values.

1. $\sum_{p=1}^{4} a_p$ 2. $\sum_{i=0}^{3} a_i$ 3. $\sum_{p=4}^{7} a_p$ 4. $\sum_{j=-2}^{1} a_j$

5. $\sum_{i=2}^{10} a_i$ 6. $\sum_{i=-5}^{4} a_i$ 7. $\sum_{j=-8}^{0} a_j$ 8. $\sum_{k=2}^{1} a_k$

*

9. Complete: $\forall_{k \geq j-1} \sum_{i=j}^{k} 1 =$

C. 1. Use the theorem in Exercise 1 of the Exploration Exercises on page 8-31 to help prove:

$$\forall_n \sum_{p=1}^{n} 6p = 3n(n + 1)$$

2. Find the sum of the first n positive multiples of 5.

D. Express each of the following in three ways in Σ-notation.

Sample. $(3 \cdot 0 + 4) + (3 \cdot 1 + 4) + (3 \cdot 2 + 4) + (3 \cdot 3 + 4)$

Solution. $\sum_{i=0}^{3}(3i + 4)$, $\sum_{p=2}^{5}(3i - 2)$, $\sum_{i=-3}^{0}(3i + 13)$

1. $(3 - 4 \cdot 2) + (3 - 4 \cdot 3) + (3 - 4 \cdot 4) + (3 - 4 \cdot 5) + (3 - 4 \cdot 6)$

2. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12}$

3. $\frac{1}{3 \cdot 4} + \frac{2}{4 \cdot 5} + \frac{3}{5 \cdot 6} + \frac{4}{6 \cdot 7}$

E. Contract into a single Σ-expression:

1. $\sum_{p=3}^{5} 2p + 12$

2. $22 + \sum_{p=8}^{11}(3p + 1)$

3. $4 + 7 + 10 + \sum_{p=8}^{12}(3p - 11) + 28$

4. $1^2 + 2^2 + 3^2 + \ldots + 10^2 - 1^2 - 2^2 - 3^2$

5. $-1 + 1 + 3 + 5 + \sum_{k=1}^{7}(2k + 5) + 44$

6. $\sum_{p=1}^{n-1} 2p(p + 2) + 2n(n + 2) + 2(n + 1)(n + 3)$

7. $\sum_{p=1}^{5}(2p^2 - 1) + 71 + 97 + \sum_{p=8}^{12}(2p^2 - 1)$

8. $\sum_{p=1}^{7}(p + 2) + \sum_{p=1}^{5}(p + 9)$

9. $\sum_{k=-5}^{4}(5k + 31) + \sum_{p=11}^{20}(5p + 1)$

10. $5 + 6 + 7 + 8 + \ldots + (n - 1) + n + (n + 1) + (n + 2)$

A DISTRIBUTIVE PRINCIPLE

In the Exploration Exercises on page 8-31 you were asked to prove the theorem:

$$\forall_x \forall_n \sum_{p=1}^{n} xa_p = x \sum_{p=1}^{n} a_p$$

In very much the same way we can prove the more general theorem:

Theorem 133.

$$\forall_x \forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} xa_i = x \sum_{i=j}^{k} a_i$$

["the left distributive principle for continued sums"]

To do so, we need a principle of mathematical induction for $\{k: k \geq j - 1\}$. This principle comes from Theorem 114 which tells us that

$$\forall_S \left[\left(j - 1 \in S \text{ and } \forall_{k \geq j-1} [k \in S \Rightarrow k + 1 \in S]\right) \Longrightarrow \forall_{k \geq j-1}\, k \in S]\right].$$

So, to construct a test-pattern for instances of Theorem 133, it is sufficient to prove:

(i) $\displaystyle\sum_{i=j}^{j-1} ca_i = c \sum_{i=j}^{j-1} a_i$ [$j - 1 \in S$]

and:

(ii) $\displaystyle\forall_{k \geq j-1} \left[\sum_{i=j}^{k} ca_i = c \sum_{i=j}^{k} a_i \Rightarrow \sum_{i=j}^{k+1} ca_i = c \sum_{i=j}^{k+1} a_i\right]$

$$\left[\forall_{k \geq j-1} [k \in S \Rightarrow k + 1 \in S]\right]$$

If you wish, go ahead and prove Theorem 133.

EXERCISES

A. 1. Use Theorems 133 and 131 to prove that, for each n, the sum of the first n even positive integers is $n(n + 1)$.

2. Use the technique for solving Exercise 1 to prove each of the following.

(a) $\forall_n \sum_{p=1}^{n} \frac{p(p+1)}{2} = \frac{n(n+1)(n+2)}{6}$ (b) $\forall_n \sum_{p=1}^{n} (p - \frac{1}{2}) = \frac{n^2}{2}$

3. Prove:

$$\forall_x \forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} x = x(k - j + 1)$$

[Hint. Use the theorem in Exercise 9 of Part B on page 8-37.]

B. Each figure shows the graph of 'y = x', for $0 \leq x \leq 1$. The segment $\overline{0, 1}$ of the x-axis has been divided into 10 congruent segments. In Figure 1, you see rectangles having these segments as bases and their upper right vertices on the line $\{(x, y): y = x\}$. In Figure 2, you see rectangles having their upper left vertices on the line.

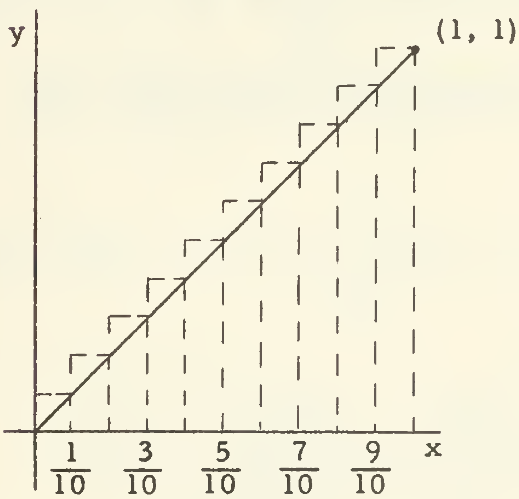


Fig. 1

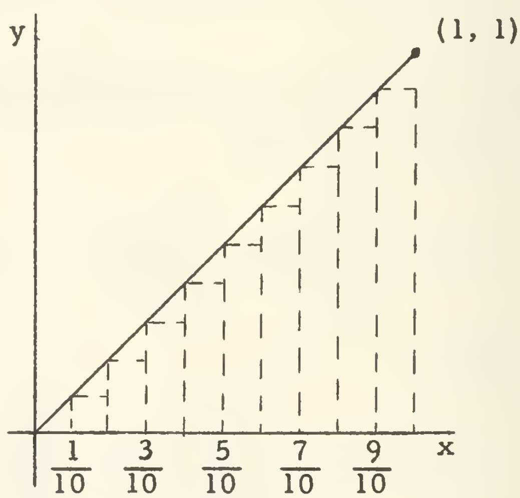


Fig. 2

1. (a) Express the sum of the area-measures of the rectangular regions in Figure 1 in Σ-notation, and use theorems you have proved to compute this sum.

 (b) Repeat (a) for the segment $\overline{0, 1}$ divided into n congruent segments. In this way, find a theorem whose 10th instance you could have used in doing (a).

2. Repeat Exercises 1(a) and (b) for Figure 2.

3. (a) The theorems you found in Exercises 1 and 2 give you ways of computing a sequence U of overestimates [Figure 1] and a sequence L of underestimates [Figure 2] of the area-measure of the triangular region whose vertices are (0, 0), (1, 0), and (1, 1). Complete the following table by computing the indicated terms of the sequences U, L, and U-L.

n	10	100	1000	10000
U_n		101/200		
L_n		99/200		
$U_n - L_n$		1/100		

(b) "Guess" the area-measure of the triangular region.

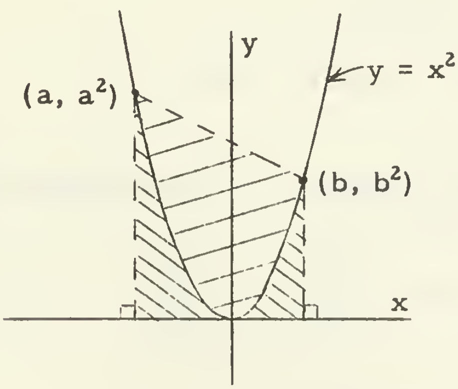
4. Repeat Exercises 1-3 with '$y = x^2$' instead of '$y = x$'.

5. Repeat Exercise 4 with '$y = x^3$' instead of '$y = x^2$'.

6. Guess:

For each m, the area-measure of the region bounded by the graphs of '$y = x^m$', '$y = 0$', and '$x = 1$' is   .

☆C. 1. Generalize the result of Exercise 4 of Part B using the interval $\overset{\longleftrightarrow}{0, a}$.

2. 

Show that the ratio of the area-measures of the shaded regions is

$$\frac{(b - a)^3}{2(b^3 - a^3)}.$$

* * *

As we have noted, Theorem 133 is an extension of the ℓdpma. Theorem 5, the sum rearrangement theorem, has a similar extension:

<u>Theorem 134.</u>

$$\forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} (a_i + b_i) = \sum_{i=j}^{k} a_i + \sum_{i=j}^{k} b_i$$

This theorem tells us, for example, that we can find the sum of the first twelve terms of the sequence whose pth term is 3-4p by adding the sums of the first twelve terms of <u>two</u> sequences. Here is how it is done.

$$\left.\begin{aligned} \sum_{p=1}^{12}(3-4p) &= \sum_{p=1}^{12}(3\cdot 1 + -4p) \\ &= \sum_{p=1}^{12} 3\cdot 1 + \sum_{p=1}^{12} -4p \end{aligned}\right\} \text{Theorem 134}$$

$$\left.\begin{aligned} &= \sum_{p=1}^{12} 3\cdot 1 + \sum_{p=1}^{12} -4p \\ &= 3\sum_{p=1}^{12} 1 + -4\sum_{p=1}^{12} p \end{aligned}\right\} \text{Theorem 133}$$

$$\left.\begin{aligned} &= 3\sum_{p=1}^{12} 1 - 4\sum_{p=1}^{12} p \\ &= 3\cdot 12 - 4\cdot \frac{12\cdot 13}{2} \end{aligned}\right\} \text{Theorem 131}$$

$$= 12(3 - 26) = 12(-23) = -276$$

In practice, one would probably skip some steps and begin with:

$$\sum_{p=1}^{12}(3-4p) = 3\sum_{p=1}^{12} 1 - 4\sum_{p=1}^{12} p$$

* * *

D. Prove Theorem 134.

E. Apply Theorem 134 in computing these sums.

1. $\sum_{p=1}^{9} (1 + 2p)$ 2. $\sum_{p=1}^{100} (p^2 + p)$ 3. $\sum_{p=1}^{50} (p^3 - p)$ 4. $\sum_{p=1}^{n} p(p + k)$

5. $\sum_{p=4}^{9} p^2$ [Hint. You will have to guess a new theorem in order to put this expression into a form to which Theorem 134 applies.]

6. Guess another new theorem which will enable you to solve Exercise 5 without using Theorem 134.

TWO TRANSFORMATION PRINCIPLES FOR CONTINUED SUMS

You probably solved Exercise 5 of Part E above as follows:

$$\sum_{p=4}^{9} p^2 = \sum_{p=1}^{6} (p + 3)^2 \qquad \text{[new theorem]}$$

$$\left.\begin{aligned} &= \sum_{p=1}^{6} (p^2 + 6p + 9) \\ &= \sum_{p=1}^{6} p^2 + 6\sum_{p=1}^{6} p + 9\sum_{p=1}^{6} 1 \end{aligned}\right\} \text{Theorems 133, 134}$$

$$\left.\begin{aligned} &= \sum_{p=1}^{6} p^2 + 6\sum_{p=1}^{6} p + 9\sum_{p=1}^{6} 1 \\ &= 91 + 6 \cdot 21 + 9 \cdot 6 \end{aligned}\right\} \text{Theorem 131}$$

$$= 271$$

You probably solved Exercise 6 of Part E as follows:

$$\sum_{p=4}^{9} p^2 = \sum_{p=1}^{9} p^2 - \sum_{p=1}^{3} p^2 \qquad \text{[new theorem]}$$

$$= 285 - 14 = 271$$

The new theorem used in Exercise 6 is the associative transformation principle for continued sums:

Theorem 135.

$$\forall_j \forall_{j_1 \geq j-1} \forall_{k \geq j_1} \sum_{i=j}^{k} a_i = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^{k} a_i$$

According to this theorem,

$$\sum_{p=1}^{9} p^2 = \sum_{p=1}^{3} p^2 + \sum_{p=3+1}^{9} p^2.$$

The following theorem is also an immediate consequence of Theorem 135 and the recursive definition of Σ-notation:

Theorem 136.

$$\forall_j \forall_{k \geq j} \sum_{i=j}^{k} a_i = a_j + \sum_{i=j+1}^{k} a_i$$

The new theorem used in Exercise 5 is the translation transformation principle for continued sums:

Theorem 137.

$$\forall_j \forall_{j_1} \forall_{k \geq j-1} \sum_{i=j}^{k} a_i = \sum_{i=j+j_1}^{k+j_1} a_{i-j_1}$$

You probably discovered this principle while solving the exercises in Parts A and D on pages 8-37 and 8-38. It tells you, for example, that

$$\sum_{p=4}^{9} p^2 = \sum_{p=4-3}^{9-3} (p+3)^2 .$$

EXERCISES

A. Complete each of the following.

Sample. $\displaystyle\sum_{p=3}^{12}(2p + 7) = \sum_{p=1}$

Solution. $\displaystyle\sum_{p=3}^{12}(2p + 7) = \sum_{p=1}^{10}\left[2(p+2)+7\right]$

1. $\displaystyle\sum_{p=5}^{10}(p - 1) = \sum_{p=1}$

2. $\displaystyle\sum_{p=5}^{10}(p - 1) = \sum_{p=7}$

3. $\displaystyle\sum_{p=5}^{10}(p - 1) = \sum_{i=0}$

4. $\displaystyle\sum_{p=5}^{10}(p - 1) = \sum_{i=-2}$

5. $\displaystyle\sum_{i=-4}^{6}(3 - 2i) = \sum_{p=1}$

6. $\displaystyle\sum_{p=1}^{n}\frac{1}{p+1} = \sum_{p=}\frac{1}{p}$

B. Compute.

1. $\displaystyle\sum_{p=1}^{14}(4p + 3)$

2. $\displaystyle\sum_{p=9}^{20}p$

3. $\displaystyle\sum_{p=1}^{17}(3 - 2p)$

4. $\displaystyle\sum_{p=8}^{19}(2 + 4p)$

5. $\displaystyle\sum_{i=-3}^{18}(i + 5)$

6. $\displaystyle\sum_{j=-5}^{12}(11 - 2j)$

7. $\displaystyle\sum_{p=4}^{15}(\tfrac{1}{2}p - 5) + \sum_{p=16}^{20}(\tfrac{1}{2}p - 5)$

8. $\displaystyle\sum_{p=1}^{50}(4 - 3p) + \sum_{q=1}^{50}(3q - 4)$

C. 1. Solve the equation:

$$\sum_{p=1}^{n} (2p + 1) = 15$$

2. Prove:

$$\forall_n \sum_{p=1}^{n} (2p - 1)^2 = \frac{n(4n^2 - 1)}{3}$$

3. Prove:

$$\forall_{k \geq -3} \sum_{i=-3}^{k} 2i = (k + 4)(k - 3)$$

4. Consider a set of n wooden cubes whose side-measures are 1, 2, 3, ..., and n, respectively. If the cubes are stacked in a step-pyramid, show that the exposed surface area-measure is $\frac{n(n + 2)(4n + 1)}{3}$.

☆5. (a) Find a set of two or more consecutive positive integers whose sum is 100. Find all possible solutions.

(b) Repeat (a) for 16 instead of 100.

☆D. For any real numbers a and b, one can define recursively a generalized Fibonacci sequence by:

$$\begin{cases} F_1 = a \\ F_2 = b \\ \forall_n \; F_{n+2} = F_n + F_{n+1} \end{cases}$$

When a = 1 = b, F is just the Fibonacci sequence f which you studied in Part E on page 8-24:

$$(*) \quad \begin{cases} f_1 = 1 \\ f_2 = 1 \\ \forall_n \; f_{n+2} = f_n + f_{n+1} \end{cases}$$

1. Write out the first ten terms of the sequence F when a = 4 and b = 6.

2. Repeat Exercise 1 with a = 1 and b = 0.

3. Compare the sequence of Exercise 2 with the sequence f.

4. Complete and prove a theorem: For any real numbers a and b,
$$\forall_n\ F_{n+2} = \quad \cdot f_n + \quad \cdot f_{n+1}$$

5. As you may have seen in solving Exercise 2, the Fibonacci sequence can be "extended backwards" by replacing the first two sentences of (*) by '$f_{-1} = 1$' and '$f_0 = 0$'. Using this extension of f, state and prove an extension of the theorem of Exercise 4:
$$\forall_n\ F_n =$$

6. State and prove a theorem for F like that of Exercise 2 of Part E on page 8-25.

☆**E.** Recall, from page 7-120, that for any positive integers m and n, the factors of n and m + n are just the factors of m and n.

1. Prove that each two consecutive terms of the Fibonacci sequence f are relatively prime.

2. Prove, for a generalized Fibonacci sequence defined in Part D, where a and b are positive integers:
$$\forall_n\ \mathrm{HCF}(F_n, F_{n+1}) = \mathrm{HCF}(a, b)$$

☆**F.** 1. Explain how Theorem 136 follows from Theorem 135 and the recursive definition of Σ-notation.

2. Prove Theorem 135. 3. Prove Theorem 137.

*

4. Prove:
$$\forall_n \sum_{p=1}^{n} (2p + 1) = (n + 1)^2 - 1$$

MISCELLANEOUS EXERCISES

1. Derive a formula for the perimeter P of a square in terms of its area-measure K.

2. Simplify.

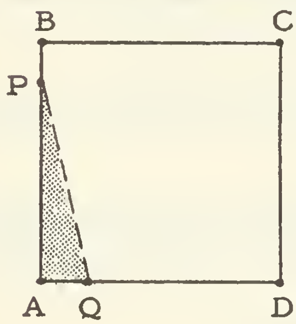
   (a) $\frac{10}{2x - y} + \frac{3}{y - 2x}$ (b) $\frac{1}{2a + 4b} + \frac{2}{a + 2b}$

3. The area of an equilateral triangle whose sides are 10 inches long is equal to the area of a trapezoid whose bases are 4 inches and 6 inches long. How long is an altitude of the trapezoid?

4. If $\frac{A}{B} = \frac{2}{3}$ then $\frac{A + B}{B} =$ ________.

5. Solve the equation '$A = \frac{BC - D}{B}$' for 'C'.

6. If a first number is twice a second, and the reciprocal of the second is 2 more than the reciprocal of the first, what are the numbers?

7. 

   Suppose that quadrilateral ABCD is a square, and that P and Q are points of the square such that $P \in \overline{BA}$, $Q \in \overline{AD}$ and $PA + AQ = AD$. Prove that the area-measure of $\triangle PAQ$ does not exceed one eighth of the area-measure of ▩ABCD.

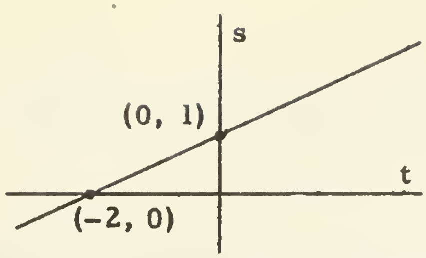
8. An apple merchant sold half of his apples at 3 for 17 cents and the other half at 5 for 17 cents. At what rate could he have sold all the apples to take in the same amount of money?

9. Solve the equation: $3^{2x+3} = 27^x$

10. Which of the given numbers is closest to $\frac{1}{3}$?

    (A) $\frac{1}{2}$ (B) $\frac{5}{16}$ (C) $\frac{4}{11}$ (D) $\frac{7}{22}$ (E) $\sqrt{0.09}$

11. Solve the equation: $\frac{5r - 3}{6} - \frac{r + 3}{2} = 1$

12. 

The variable quantity s is a linear function of the variable quantity t. The graph at the left shows a plotted against t. What value of s corresponds with the value 20 of t?

13. Derive a formula for the area-measure K of a rectangle in terms of its perimeter P and its width-measure w.

14. The number of diagonals of a polygon with n sides is $(n^2 - 3n)/2$. If a polygon has 90 diagonals, how many sides does it have?

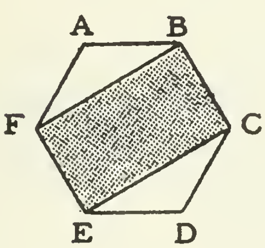
15. In an election, the winner had a majority of 106 votes over the loser. If 1528 votes were cast, how many did the loser receive?

16. Simplify.

(a) $\sqrt{\frac{x^2}{9} + \frac{x^2}{16}}$ (b) $4\frac{5}{8} - 2\frac{3}{4} \div 1\frac{1}{4}$ (c) $\frac{1}{2p} + \frac{2}{3p^2}$

17. Solve these equations.

(a) $400 = x\%(12000) + 100$ (b) $x - 20\%(x) = 8.08$

18. 

What per cent of the regular hexagonal region ABCDEF is shaded?

19. Simplify.

(a) $\dfrac{2 + \dfrac{6a}{a + 3b}}{\dfrac{3a}{a + 3b}}$ (b) $7t - \dfrac{8}{7t - \dfrac{8}{7t}}$

20. Solve: $3^x = 9^2$

21. If $x - y = 24$ and $xy = 12$ then $\frac{1}{x} - \frac{1}{y} =$.

22. If the product of two positive numbers is 4 and the sum of their squares is 9, what is the ratio of their product to their sum?

23. If the average weight of a 5-man team is x pounds and the lightest man weighs y pounds then the average weight of the other players is   pounds.

24. Solve the equation '$x = \frac{y}{1 - z}$' for 'z'.

25. Suppose that a, b, and c are the measures of the edges of a rectangular solid. Show that if x, y, and z are the measures of the edges of a second rectangular solid such that $x < a$, $y < b$, and $z < c$ and such that the surface area-measure of the second solid is half that of the first, then the volume-measure of the second is not half that of the first.

26. A rocket traveling at 3300 feet per second is moving at the rate of how many miles per hour?

27. The vertices of a triangle are $(2, -1)$, $(5, 4)$, and $(-3, 2)$. Prove that the triangle is a right isosceles triangle.

28. A ladder 36 feet long leans against a wall. It makes an angle of 63° with the ground. How far is the foot of the ladder from the foot of the wall, assuming that the ground is level?

29. What is the cost of n shirts if $4\frac{1}{2}$ dozen cost g dollars?

30. If a circular region and a square region have the same center and the same area-measure, the length of that portion of the square region's boundary which is outside the circle is what per cent of the length of its total boundary?

31. If a man travels x miles per hour for y hours and then u miles per hour for v hours, what is his average rate for the entire trip?

32. Simplify.

(a) $\frac{2}{5}(x-y) \cdot \frac{5}{7}(x+y)$ (b) $\frac{8}{9}(y+2) \cdot \frac{3}{4}(y-3)$ (c) $(\frac{1}{3}ab)^2(\frac{3}{2ab^2})^3$

33. Suppose that A, B, and C are the vertices of a triangle. If quadrilateral APQR is a rhombus such that $P \in \overline{AB}$, $Q \in \overline{BC}$, and $R \in \overline{CA}$, and AB = 18, BC = 16, and AC = 24, find AP, BQ, and CR.

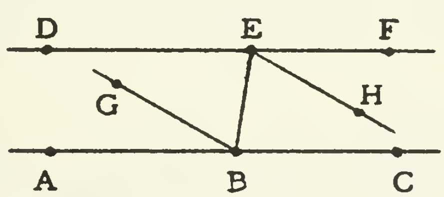
34. A team won n games, lost m games, and tied p games. What per cent of games played did it win?

35. How many members has

$\{(x, y) \in I \times I: y = x\} \cap \{(x, y): |x - 2| < y - 1 \text{ and } |x - 3| < 2 - \frac{y}{2}\}$?

36. Solve this equation: $(a + 3)^2 + (a + 3) - 2 = 0$

37. A right circular cylinder has radius 6 and altitude 2. By what number can you increase either the radius <u>or</u> the altitude to produce the same volume change?

38. 

If $\overleftrightarrow{DF} \parallel \overleftrightarrow{AC}$, $\overleftrightarrow{EH} \parallel \overleftrightarrow{GB}$, $\angle DEB$ is an angle of 80°, and $\angle HEB$ is an angle of 70°, then $°m(\angle GBA) =$ ______ .

39. Simplify.

(a) $(6x^3 - 5x - 3) + (8x^3 + x^2 - 5) - (3x^3 - 2x^2 + x - 1)$

(b) $[8(x + y) - 2(x + y)^2 - 5] + [3(x + y)^3 - 2(x + y) + 7]$

(c) $24x(\frac{1}{12}x^2 - \frac{5}{6}x + 1) - 36x(-\frac{1}{9}x^2 + \frac{2}{3}x - \frac{1}{6})$

40. Complete.

(a) $\forall_x\ x^3 - 125 = (x - 5)(\qquad)$ (b) $\forall_x\ x^6 - 1 = (x - 1)(x + 1)(\qquad)$

41. Derive formulas for the perimeter P and the area-measure K of a regular hexagon in terms of its diameter d.

SUMMATION BY DIFFERENCES

Here is a summation theorem which you can establish in a number of ways:

$$(*) \qquad \forall_n \sum_{p=1}^{n} (2p + 1) = (n + 1)^2 - 1$$

[Suggest two ways of proving this theorem.] This theorem, and many others, can be discovered and proved by a method involving a clever bit of algebra. Here is how such a proof of (*) goes:

$$\sum_{p=1}^{q} (2p + 1) = \sum_{p=1}^{q} [(p + 1)^2 - p^2] \qquad \text{[Check this algebra.]}$$

$$= \sum_{p=1}^{q} (p + 1)^2 - \sum_{p=1}^{q} p^2$$

$$= \sum_{p=2}^{q+1} p^2 - \sum_{p=1}^{q} p^2$$

$$= \left(\sum_{p=2}^{q} p^2 + (q + 1)^2\right) - \left(1^2 + \sum_{p=2}^{q} p^2\right)$$

$$= (q + 1)^2 - 1^2$$

Hence,

$$\forall_n \sum_{p=1}^{n} (2p + 1) = (n + 1)^2 - 1.$$

Notice that the important step in the proof is the very first one in which each term of the given sequence is expressed as the difference between a pair of <u>consecutive</u> terms of another sequence.

In general, to discover a summation theorem for a sequence b:

$$\forall_n \sum_{p=1}^{n} b_p = ?$$

we try to find a sequence a such that, for each p,

$$b_p = a_{p+1} - a_p.$$

Once we have such a sequence a, the same technique establishes that

$$\forall_n \sum_{p=1}^{n} b_p = a_{n+1} - a_1.$$

In fact, this technique proves:

<u>Theorem 138.</u>

$$\forall_n \sum_{p=1}^{n} (a_{p+1} - a_p) = a_{n+1} - a_1$$

[Although we have indicated a technique for proving Theorem 138, the easiest proof is by induction. You can work that proof out in your head.]

EXERCISES

<u>A.</u> Use Theorem 138 to compute each of the continued sums.

1. $\sum_{p=1}^{50} (\frac{1}{p+1} - \frac{1}{p})$

2. $\sum_{p=1}^{50} [(p+1) - p]$

3. $\sum_{p=1}^{50} (\frac{1}{p} - \frac{1}{p+1})$

4. $\sum_{p=1}^{48} (\sqrt{p+1} - \sqrt{p})$

5. $\sum_{p=1}^{50} \big[[3(p+1) + 7] - [3p + 7]\big]$

6. $\sum_{p=1}^{50} [(4p + 11) - (4p + 7)]$

7. $\sum_{p=1}^{50}[(2p + 1) - (2p - 1)]$

8. $\sum_{p=1}^{50}\left(\frac{1}{2p + 1} - \frac{1}{2p - 1}\right)$

9. $\sum_{p=1}^{50}\left(\frac{p + 1}{p + 4} - \frac{p}{p + 3}\right)$

10. $\sum_{p=6}^{29}\left(\frac{1}{\sqrt{p - 5}} - \frac{1}{\sqrt{p - 4}}\right)$

11. $\sum_{p=1}^{50}[p(p - 1) - (p - 1)(p - 2)]$

12. $\sum_{p=1}^{50}[(p + 2)(p + 3) - (p + 3)(p + 4)]$

*

Complete.

13. $\forall_n \sum_{p=1}^{n}\left(\frac{p + 3}{p + 6} - \frac{p + 2}{p + 5}\right) =$

14. $\forall_n \sum_{p=1}^{n}\left(\frac{1}{3p - 1} - \frac{1}{3p + 2}\right) =$

15. $\forall_n \sum_{p=1}^{n}(\sqrt{p + 1} - \sqrt{p}) =$

16. $\forall_m \forall_{n \geq m} \sum_{p=m}^{n}(\sqrt{p + 1} - \sqrt{p}) =$

17. $\forall_n \sum_{p=1}^{n} \frac{\sqrt{p + 1} - \sqrt{p}}{\sqrt{p(p + 1)}} =$

18. $\forall_n \sum_{p=1}^{n} \frac{1}{\sqrt{p + 1} + \sqrt{p}} =$

[Hint for Exercise 18. Prove: $\forall_p \frac{1}{\sqrt{p + 1} + \sqrt{p}} = \sqrt{p + 1} - \sqrt{p}$]

B. 1. Use this summation theorem [equivalent to the one proved on page 8-52]:

$$\forall_n \sum_{p=1}^{n}(2p + 1) = n^2 + 2n$$

to rediscover the theorem:

$$\forall_n \sum_{p=1}^{n} p = \frac{n(n + 1)}{2}$$

[Hint. $\Sigma(2p + 1) = 2\Sigma p + \Sigma 1$]

2. The algebra theorem:

$$\forall_p\ (p + 1)^2 - p^2 = 2p + 1$$

and Theorem 138 led us, in Exercise 1 above, to a new derivation of the summation theorem:

$$\forall_n \sum_{p=1}^{n} p = \frac{n(n + 1)}{2}$$

Complete the following algebra theorem:

$$\forall_p\ (p + 1)^3 - p^3 =$$

and use it to get a new derivation of the summation theorem:

$$\forall_n \sum_{p=1}^{n} p^2 = \frac{n(n + 1)(2n + 1)}{6}$$

3. Complete the following algebra theorem:

$$\forall_p\ p^3 - (p - 1)^3 =$$

and use it instead of the one in Exercise 2 to derive the summation theorem for the sequence of squares. [<u>Hint</u>. In this case, in using Theorem 138, $\forall_p\ a_p = (p - 1)^3$.]

☆4. Use the procedure suggested in Exercises 2 or 3 to sum the first n terms of the sequence of fourth powers.

<u>C</u>. 1. (a) Prove the algebra theorem:

$$\forall_p\ p(p - 1) - (p - 1)(p - 2) = 2(p - 1)$$

(b) Use this theorem to prove:

$$\forall_n \sum_{p=1}^{n} (p - 1) = \frac{n(n - 1)}{2} \qquad \text{[Theorem 139a]}$$

2. (a) Prove the algebra theorem:

$$\forall_p \; p(p-1)(p-2) - (p-1)(p-2)(p-3) = 3(p-1)(p-2)$$

(b) Use this theorem to prove:

$$\forall_n \sum_{p=1}^{n} (p-1)(p-2) = \frac{n(n-1)(n-2)}{3} \qquad \text{[Theorem 139b]}$$

3. Compare the summation theorems proved in Exercise 1(b) and in Exercise 2(b). State and prove the next two theorems of this type. [Theorems 139c and d]

☆<u>D</u>. 1. Suppose we wish to apply Theorem 138 to rediscover the summation theorem:

$$\forall_n \sum_{p=1}^{n} \frac{1}{p(p+1)} = \frac{n}{n+1}$$

After some experimenting, we find that, for each p,

$$\frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}.$$

So, if for each p, $a_p = \frac{1}{p}$,

$$\frac{1}{p(p+1)} = a_p - a_{p+1}.$$

This sequence a is not quite what we were looking for. Why? But, the situation is easily remedied. Remedy it, and find the required summation theorem.

2. Rediscover the summation theorem:

$$\forall_n \sum_{p=1}^{n} \frac{1}{(2p-1)(2p+1)} =$$

[<u>Hint</u>. Exercise 1 might suggest that, for each p,

$$\frac{1}{(2p-1)(2p+1)} = \frac{1}{2p-1} - \frac{1}{2(p+1)-1}.$$

This is not so. However, it is easily corrected.]

3. Guess and prove a summation theorem:

$$\forall_n \sum_{p=1}^{n} \frac{1}{(3p - \quad)(3p+1)} =$$

[Hint. If, for each p, $a_{p+1} = \frac{1}{3p+1}$ then, for each p, $a_p = \frac{1}{3p - \qquad}$.]

4. Guess and prove a summation theorem:

$$\forall_m \forall_n \sum_{p=1}^{n} \frac{1}{[mp - (\qquad)](mp+1)} =$$

5. There are still more general theorems than the one you guessed in Exercise 4. Try to find one. [Hints. (1) Did you use the fact that $m \in I^+$? (2) In each of the theorems in the exercises above, there is an expression of the form '. . . $\cdot$p + 1'. In proving these theorems did you use any special property of 1?]

DIFFERENCE-SEQUENCES

Consider a sequence a whose first ten terms are

10 18 28 40 54 70 88 108 130 154.

From this sequence we can form a new sequence, its difference-sequence, called 'Δa'.

8 10 12 14 16 18 20 22 24

Note that, for each p,

$$(\Delta a)_p = a_{p+1} - a_p.$$

Suppose that b is the sequence such that $\forall_n\, b_n = 3 + 2n^2$. What is the fourth term of the difference-sequence for b? What is $(\Delta b)_5$? What is $(\Delta b)_{150}$?

Consider the sequence c such that $\forall_n\, c_n = 2(2n+1)$. What is c_4? c_5? c_{150}? Is it the case that

$\forall_n\, (\Delta b)_n = c_n$? [Justify your answer.]

EXERCISES

A. For each given sequence, find as many terms as you can for its indicated difference-sequences.

1. a: 24 36 50 66 84 104 126 150

   Δa: 12

2. c: −6 −6 −4 0 6 14 24 36

   Δc:

3. s: 2 12 36 80 150 252 392

   Δs:

4. t: 6 16 40 84 154 256 396

   Δt:

5. u: 4 24 72 160 300 504 784

   Δu:

6. k: −3 −1 11 39 89 167 279

   Δk: 2

   $\Delta(\Delta k)$: 10

7. s: 0 1 4 9 16 25 36 49 64 81

   Δs:

   $\Delta(\Delta s)$:

8. c: 0 1 8 27 64 125 216 343 512

   Δc:

   $\Delta(\Delta c)$:

   $\Delta^3 c$: 6

9. s: a

   Δs: m

   $\Delta^2 s$: n

   $\Delta^3 s$: k k k k

B. In each exercise you are given the first several terms of a sequence. Find as many terms as you can of some sequence whose difference-sequence is the given one.

1. a:
   Δa: 3 5 8 12 17 23 30 38

2. v:
   Δv: 8 9 10 11 12 13 14 15

3. x:
   Δx: 3 3 3 3 3 3 3 3

4. y:
   Δy: -2 -2 -2 -2 -2 -2 -2 -2

5. z:
   Δz:
   $\Delta^2 z$: 5 4 3 2 1 0 -1

C. How many sequences are there which have a given sequence as difference-sequence? How many with a given first term?

EXPLORATION EXERCISES

For each exercise, you are given a number and a sequence. Find the fiftieth term of the sequence whose first term is the given number, and whose difference-sequence is the given sequence.

1. a: 4
   Δa: 1 2 3 4 ... p ...

2. $a_1 = 7, \quad \forall_p \, (\Delta a)_p = p$

3. b: 1
   Δb: 3 3 3 3 ... 3 ...

4. $b_1 = 8, \quad \forall_p \; (\Delta b)_p = 2, \quad b_{50} =$

5. $t_1 = 2, \quad \forall_p \; (\Delta t)_p = p - 1, \; t_{50} =$

6. $s_1 = 7, \quad \forall_p \; (\Delta s)_p = (p - 1)(p - 2), \quad s_{50} =$

See Part C on page 8-55.

USING DIFFERENCE-SEQUENCES IN SUMMING

Undoubtedly you discovered in the Exploration Exercises that, for any sequence a,

Theorem 140.

$$\forall_n \; a_n = a_1 + \sum_{p=1}^{n-1} (\Delta a)_p$$

This is a corollary of Theorem 138. Prove it.

Theorem 140 is helpful in summing a sequence when you are given just the first term of a sequence and its difference-sequence. For example, suppose that

$a_1 = 11$ and, for each p, $(\Delta a)_p = 3p + 2$.

Our problem is to find the summation theorem for the sequence a:

$$\forall_n \sum_{p=1}^{n} a_p = \;?$$

We can use Theorem 140 to find the qth term of a:

$$a_q = a_1 + \sum_{p=1}^{q-1} (\Delta a)_p$$

$$= 11 + \sum_{p=1}^{q-1} (3p + 2)$$

$$= 11 + 3\frac{(q-1)q}{2} + 2(q-1)$$

$$= \frac{1}{2}(3q^2 + q + 18)$$

So,

$$\sum_{p=1}^{q} a_p = \frac{1}{2}\sum_{p=1}^{q}(3p^2 + p + 18)$$

$$= \frac{1}{2}\left[3\frac{q(q+1)(2q+1)}{6} + \frac{q(q+1)}{2} + 18q\right]$$

$$= \frac{q(q^2 + 2q + 19)}{2}.$$

Hence,

$$\forall_n \sum_{p=1}^{n} a_p = \frac{n(n^2 + 2n + 19)}{2}.$$

Let's consider a more complicated example.

$$\left.\begin{array}{lllllll} a: & 10 & & & & & \\ \Delta a: & & -3 & & & & \\ \Delta^2 a: & & & 5 & & & \\ \Delta^3 a: & & & & 7 \quad 7 & \ldots \quad 7 & \ldots \end{array}\right\} \forall_n \sum_{p=1}^{n} a_p = \;?$$

Theorem 140 will help us find the qth term of $\Delta^2 a$. Knowing this, we can use Theorem 140 again to find the qth term of Δa, and knowing this we can find the qth term of a. With this, we can find the required summation theorem.

$$(\Delta^2 a)_q = (\Delta^2 a)_1 + \sum_{p=1}^{q-1}(\Delta^3 a)_p$$

$$= 5 + \sum_{p=1}^{q-1} 7$$

$$= 5 + 7(q-1)$$

Now that we know the qth term of $\Delta^2 a$, we find the qth term of Δa.

$$(\Delta a)_q = (\Delta a)_1 + \sum_{p=1}^{q-1} (\Delta^2 a)_p$$

$$= -3 + \sum_{p=1}^{q-1} [5 + 7(p - 1)]$$

$$= -3 + 5(q - 1) + 7\frac{(q - 1)(q - 2)}{2} \qquad \text{[Theorem 139a]}$$

And, now that we know the qth term of Δa, we can find the qth term of a.

$$a_q = a_1 + \sum_{p=1}^{q-1} (\Delta a)_p$$

$$= 10 + \sum_{p=1}^{q-1} [-3 + 5(p - 1) + 7\frac{(p - 1)(p - 2)}{2}]$$

$$= 10 + -3(q - 1) + 5\frac{(q - 1)(q - 2)}{2} + 7\frac{(q - 1)(q - 2)(q - 3)}{2 \cdot 3}$$

Finally,

$$\sum_{p=1}^{n} a_p = \sum_{p=1}^{n} [10 - 3(p - 1) + 5\frac{(p - 1)(p - 2)}{2} + 7\frac{(p - 1)(p - 2)(p - 3)}{2 \cdot 3}]$$

$$= 10n - 3\frac{n(n - 1)}{2} + 5\frac{n(n - 1)(n - 2)}{2 \cdot 3} + 7\frac{n(n - 1)(n - 2)(n - 3)}{2 \cdot 3 \cdot 4}.$$

[Of course, if you had much use for this last expression you might want to simplify it.]

EXERCISES

A. 1. $a_1 = 3$, $(\Delta a)_1 = 5$, $\forall_p\ (\Delta^2 a)_p = 7$, $a_{10} =$

2. $a_1 = 5,\ (\Delta a)_1 = 7,\ \forall_p\ (\Delta^2 a)_p = 11,\ \sum_{p=1}^{10} a_p =$

3. $a_1 = -11,\ (\Delta a)_1 = 7,\ (\Delta^2 a)_1 = 5,\ \forall_p\ (\Delta^3 a)_p = 17,\ a_{20} =$

4. $a_1 = 1,\ (\Delta a)_1 = 5,\ \forall_p\ (\Delta^2 a)_p = p,\ \forall_n \sum_{p=1}^{n} a_p =$

5. $a_1 = 7,\ \forall_p\ (\Delta a)_p = d,\ \forall_n\ a_n = \qquad ,\ \forall_n \sum_{p=1}^{n} a_p =$

B. In the sequence

$$7, \quad 18, \quad 29, \quad 40, \quad 51, \quad 62, \quad \ldots$$

each term is 11 less than the succeeding term. Find the 78th term, and the sum of the first 100 terms.

C. Consider the familiar problem of finding the sum of the cubes of the first n positive integers. Let's see if we can apply the method of difference-sequences.

$$\begin{array}{lccccccccccccc}
a: & 1 & & 8 & & 27 & & 64 & & 125 & & 216 & & \ldots \\
\Delta a: & & 7 & & 19 & & 37 & & 61 & & 91 & & \ldots & \\
\Delta^2 a: & & & 12 & & 18 & & 24 & & 30 & & \ldots & & \\
\Delta^3 a: & & & & 6 & & 6 & & 6 & & \ldots & & &
\end{array}$$

We might guess from this limited evidence that $\Delta^3 a$ is a <u>constant</u> sequence. [Remember that a constant function, or a constant, is one which has only one value.]

1. We can partially check this idea by assuming that the next term in $\Delta^3 a$ is 6, and then work backwards to find the next terms in $\Delta^2 a$, Δa, and a. If the last turns out to be 7^3, we have more evidence. Check it.

2. Of course, this manner of checking does not prove that the 3rd difference-sequence for the sequence of cubes is the constant 6. But, as follows from the recursive definition of Σ-notation and the definition of Δ, given any sequences a and b and any number c,

$$\forall_n \; a_n = c + \sum_{p=1}^{n-1} b_n \Longrightarrow (a_1 = c \text{ and } \forall_n (\Delta a)_n = b_n).$$

In other words, given a number c and a sequence b, the sequence a which you find by the method of difference-sequences [assuming that $a_1 = c$ and $\Delta a = b$] is the one and only sequence whose first term is c and whose first difference-sequence is b. Consequently [since $\Delta^3 a$ is the first difference-sequence of $\Delta^2 a$, and $\Delta^2 a$ is the first difference-sequence of Δa], if we can show by the method of difference-sequences that

$$[a_1 = 1, (\Delta a)_1 = 7, (\Delta^2 a)_1 = 12, \Delta^3 a = 6] \Longrightarrow \forall_p \; a_p = p^3$$

then it will follow that the 3rd difference-sequence of the sequence of cubes <u>is</u> the constant 6. Do this.

3. Continue part of the work you did in solving Exercise 2 to complete:

$$\forall_n \sum_{p=1}^{n} p^3 = n + 7\frac{n(n-1)}{2} +$$

4. (a) Suppose that a is the sequence of sixth powers of the positive integers. Do you think that one of the successive difference-sequences of a is a constant? If so, which is the first one which is constant?

   (b) Complete:

$$\forall_n \sum_{p=1}^{n} p^6 = n + 63\frac{n(n-1)}{2} +$$

5. (a) Construct a sequence other than the sequence of cubes for which the third difference-sequence is the first one which is a constant.

   (b) Give a formula for the sum of the first n terms of your sequence.

6. Complete: For any numbers r, s, t, and w,

(a) $[a_1 = s,\ \Delta a = r] \Longleftrightarrow \left[\forall_n\ a_n = s + \sum_{p=1}^{n-1} (\Delta a)_p = s + r(\qquad)\right]$

(b) $[a_1 = t,\ (\Delta a)_1 = s,\ \Delta^2 a = r] \Longleftrightarrow \left[a_1 = t,\ \forall_n\ (\Delta a)_n = \qquad\right]$

$\Longleftrightarrow \left[\forall_n\ a_n = \qquad\right]$

(c) $[a_1 = w,\ (\Delta a)_1 = t,\ (\Delta^2 a)_1 = s,\ \Delta^3 a = r] \Longleftrightarrow \left[\forall_n\ a_n = \qquad\right]$

D. An object was thrown downward. At the end of each tenth-second the distance the body had fallen was recorded. [d_j ($j = 0, 1, 2, ---, 6$) is the distance fallen at the end of the jth tenth-second.]

j	0	1	2	3	4	5	6
d_j	0	0.46	1.24	2.35	3.77	5.5	7.55
$(\Delta d)_j$	0.46	0.78					
$(\Delta^2 d)_j$	0.32						

1. Complete the table.

2. Although $\Delta^2 d$ is not a constant, it is nearly enough so to suggest that the variation is due to errors in measurement. Use the technique of difference-sequences to obtain a formula for d_j on the assumption that $\Delta^2 d = 0.32$.

3. Compare the values given by your formula with the given data.

4. In doing Exercise 3 you should have found that your computed values agree quite well with the data. Supposing that the object continued falling, how far do you think it would fall during the first second? How far do you think the object fell during the first 1/3 second? [Why 'think'?]

5. Suppose that s_t is the distance the body falls in t seconds. Find a formula for s_t.

ARITHMETIC PROGRESSIONS

A sequence whose <u>first</u> difference-sequence is a constant is called <u>an arithmetic progression</u> [AP]. For an arithmetic progression a, the value of the constant sequence Δa is called <u>the common difference</u> of a.

Here are lists of seven numbers. For each list, there are many sequences whose first seven terms are, in the given order, the numbers listed. In some cases, one of these sequences is an AP. For each such case, tell the common difference of the AP and its eighth term.

(1) 5, 7, 9, 11, 13, 15, 17

(2) −3, 1, 5, 9, 13, 17, 21

(3) 4, 5, 7, 10, 14, 19, 25

(4) 3, 6, 12, 24, 48, 96, 192

(5) π, 3π, 5π, 7π, 9π, 11π, 13π

(6) 8, 1, −6, −13, −20, −27, −34

(7) −1, 0, 2, 6, 14, 30, 62

(8) 1, 2, 4, 8, 16, 32, 64

EXERCISES

<u>A</u>. Fill the blanks to get arithmetic progressions.

1. −2, −1, , 1, 2, , , ...
2. 3, , , 66, , , ...
3. 9, 10, 11, , , , , ...
4. −8, , , , , , 17, ...
5. , , , 16, , 24, ...
6. , , , , , −30, −35, ...
7. 6, , , , , , 7, ...
8. $\frac{1}{3}$, $\frac{5}{6}$, , , , , ...
9. 3, , 7, , 9, ...
10. −3, , $-3 + 4\sqrt{2}$, , , , ...
11. , , , , 7, , , ...

<u>B</u>. 1. If the first term of an AP is 4 and the common difference is 10, what is the twelfth term?

2. Suppose that a sequence a is an AP. If $a_1 = 3$ and $\Delta a = -4$, what is a_{11}?

3. Theorem 140 tells you that, for any sequence a at all,

$$\forall_n \; a_n = a_1 + \sum_{p=1}^{n-1} (\Delta a)_p .$$

Use this theorem to derive a formula for the nth term of an AP whose first term is a_1 and whose common difference is d.

4. Suppose that a is an AP with common difference d.

Prove: $\forall_n \forall_{m \neq n} \; d = \dfrac{a_m - a_n}{m - n}$ [Theorem 141b]

C. Find the indicated term of the given AP.

Sample. 12, 15, 18, ...; 17th term

Solution. Exercise 3 of Part B gives us a formula for finding the nth term of an AP:

$$a_n = a_1 + (n - 1)d$$ [Theorem 141a]

So, in this case, since $a_1 = 12$, and $d = 3$,

$$a_{17} = 12 + (17 - 1)3 = 60 .$$

1. 5, 11, 17, ...; 29th term
2. 59, 57, ...; 15th term
3. 17, ..., −3, −5, ...; 4th term
4. 9, $\frac{21}{2}$, 12, ...; 16th term
5. c − 2d, 2c − 3d, 3c − 4d, ...; 7th term
6. $a_{20} = 14$, $a_{24} = -6$; 13th term

D. 1. Use the method of difference-sequences to prove that, for any arithmetic progression a,

$$\forall_n \sum_{p=1}^{n} a_p = \frac{n}{2}[2a_1 + (n - 1)d].$$ [Theorem 141c]

2. Prove: For any AP a, $\forall_n \sum_{p=1}^{n} a_p = \frac{n}{2}(a_1 + a_n)$ [Theorem 141d]

E. Find the sum of the given number of successive terms of the given AP, starting with the first.

Sample. 30 terms of 12, 15, 18, ...

Solution. The theorem proved in Exercise 1 of Part D gives us a formula for finding the sum of the first n terms of an AP:

$$s_n = \frac{n}{2}[2a_1 + (n - 1)d]$$ [Theorem 141c]

So, in this case,

$$s_{30} = \frac{30}{2}[2 \cdot 12 + 29 \cdot 3] = 1665.$$

1. 20 terms of $-2, 5, 12, \ldots$
2. 12 terms of $-100, 0, 100, \ldots$
3. 7 terms of $1, 0.5, 0, \ldots$
4. 50 terms of $-3, -2.5, -2, \ldots$
5. 12 terms of $9, 9, 9, \ldots$
6. 19 terms of $2, 4, 6, \ldots$
7. 1000 terms of $1, 3, 5, \ldots$
8. 28 terms of $\sqrt{2},\ 3 + \sqrt{2},\ 6 + \sqrt{2}, \ldots$
9. 1001 terms of $1, -4, -9, \ldots$
10. 1001 terms of $1, \frac{999}{1001}, \frac{997}{1001}, \ldots$

*

11. Twenty successive terms starting with the fifth term of

$$7, 11, 15, \ldots$$

12. Forty-eight successive terms starting with the forty-eighth term of $64, 58, 52, \ldots$

13. Find the sum of the first thousand positive integers by using Exercise 2 of Part D.

F. 1. Solve Problem I on page 7-1 of Unit 7.

2. Find the sum of all positive integers less than 200 which are divisible by 3.

3. Find the sum of all positive integers less than 200 which are divisible by neither 3 nor 5.

4. A job printer charges \$5 for printing the first 100 programs for a concert. Each additional hundred costs 50 cents less than the preceding until a minimum rate of \$2 per hundred is reached. How much would he charge for printing 1000 programs?

5. Divide \$4000 among 8 children in a family in such a way that the youngest child gets \$20 less than the next older, the latter gets \$20 less than the next older, etc.

6. (a) A grocer stacks canned goods in a triangular display like the one shown in the figure on the right. How many cans are there in a 10-high stack?

(b) If the cans are shipped 24 to a case, what is the smallest number of cases he needs to make a triangular stack, assuming that he wishes to use all the cans in the cases? If such a stack is possible, how many rows does it have?

7. As you learned in Unit 5, if f and g are functions such that $\mathcal{R}_g \subseteq \mathcal{D}_f$ then $f \circ g$ is a function whose domain is $\mathcal{D}_g$. In particular, if the terms of a sequence a belong to the domain of a function f then $f \circ a$ is also a sequence. Suppose that a is an AP with common difference d. For each function given below, tell if the result of composing it with a is an AP, and if it is, give the common difference.

(a) $f(x) = x + 5$ (b) $g(x) = 7x$

(c) $h(x) = x^2$ (d) $k(x) = 3x + 2$

(e) $s(x) = 9$ (f) $t(x) = -3x + 1$

(g) $u(x) = \frac{1}{2}x - 1$ (h) $v(x) = \frac{1}{x}$

8. The sum of three consecutive terms of an AP is 27, and the product is 504. Find the terms.

9. Suppose that a is an AP with common difference d and that a′ is an AP with common difference d′.

   (a) Is a + a′ an AP? If so, what is its common difference?

   (b) Under what conditions will $a \circ a'$ be an AP?

10. Prove that, for each m and n, n is the sum of m consecutive positive integers if and only if $\frac{2n}{m} - m$ is an odd positive integer.

G. The following exercises refer to the equations:

(a) $s_n = \frac{n}{2}[2a_1 + (n-1)d]$ (b) $s_n = \frac{n}{2}(a_1 + a_n)$

1. Solve equation (a) for 'a_1'.

2. Solve equation (a) for 'd'.

3. Find the common difference of an AP whose first term is 10 and the sum of whose first 10 terms is 98.

4. Find the first term of an AP whose common difference is $\frac{1}{2}$ and the sum of whose first 30 terms is $\frac{1}{2}$.

5. Find the first term and common difference of an AP the sum of whose first 15 terms is 100 and the sum of whose first 20 terms is 130.

6. Given an AP with first term $-\frac{3}{2}$ and common difference 2. How many terms must be added, starting with the first, to get the sum 350?

7. Is there an AP with first term 2 and common difference −4 such that, for some n, the sum of the first n terms is −37?

8. Solve equation (b) for 'a_n'.

9. Solve equation (b) for 'a_1'.

10. What is the 10th term of an AP whose first term is 1 and the sum of whose first 10 terms is 5?

11. What is the average of the first and eleventh terms of an AP the sum of whose first eleven terms is 24? What is its first term?

H. 1. Find the sum of the first 16 terms of the AP

(a) whose first term is 7 and whose common difference is -8;

(b) whose first term is $4\sqrt{2}$ and whose common difference is $-\sqrt{2}$;

(c) whose fifth term is 9 and whose eighth term is 10;

(d) whose common difference is 5 and the sum of whose first 100 terms is 100;

(e) the sum of whose first 10 terms equals the sum of its first 6 terms.

☆2. Assuming that the sequence a is an AP, prove:

$$\forall_m \forall_n \left[\left(m \neq n \text{ and } \sum_{p=1}^{m} a_p = \sum_{p=1}^{n} a_p\right) \Longrightarrow \sum_{p=1}^{m+n} a_p = 0\right]$$

☆I. In the following exercises, a is an AP with common difference d. Assume, also, that the terms of a are positive. Prove the theorems in these exercises.

1. $\forall_n \sum_{p=1}^{n} \frac{1}{a_p a_{p+1}} = \frac{n}{a_1 a_{n+1}}$ [Hint. Recall Part D on page 8-56.]

2. $\forall_n \sum_{p=1}^{n} \frac{1}{\sqrt{a_p} + \sqrt{a_{p+1}}} = \frac{n}{\sqrt{a_1} + \sqrt{a_{n+1}}}$

[Hint. First, prove: $\forall_{x \geq 0} \forall_{0 \leq y \neq x} \frac{1}{\sqrt{x} + \sqrt{y}} = \frac{\sqrt{x} - \sqrt{y}}{x - y}$]

* * *

☆There is another way of proving that, for any arithmetic progression,

$$(*) \qquad \forall_n \sum_{p=1}^{n} a_p = \frac{n}{2}[2a_1 + (n-1)d].$$

The idea of the proof can be shown schematically, as follows:

$$\begin{array}{rl}
s_n = & a_1 + [a_1 + d] + \ldots + [a_1 + (n-2)d] + [a_1 + (n-1)d] \\
s_n = & [a_1 + (n-1)d] + [a_1 + (n-2)d] + \ldots + [a_1 + d] + a_1 \\
\hline
2s_n = & [2a_1 + (n-1)d] + [2a_1 + (n-1)d] + \ldots + [2a_1 + (n-1)d] + [2a_1 + (n-1)d] \\
= & n[2a_1 + (n-1)d] \\
s_n = & \frac{n}{2}[2a_1 + (n-1)d]
\end{array}$$

To give a formal proof of (*) along these lines, we need a new theorem--<u>the reflection transformation principle for continued sums</u>:

<u>Theorem 142.</u>

$$\boxed{\forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} a_i = \sum_{i=j}^{k} a_{k+j-i}}$$

This theorem tells us, for example, that

$$\sum_{p=1}^{2} a_p = \sum_{p=1}^{2} a_{3-p} \qquad [a_1 + a_2 = a_2 + a_1],$$

and that

$$\sum_{p=1}^{3} a_p = \sum_{p=1}^{3} a_{4-p} \qquad [a_1 + a_2 + a_3 = a_3 + a_2 + a_1].$$

* * *

☆<u>J</u>. 1. Use Theorem 142 to prove:

$$\forall_n \sum_{p=1}^{n} [a_1 + (p-1)d] = \sum_{p=1}^{n} [a_1 + (n-p)d]$$

2. Use the result of Exercise 1 in proving (*).

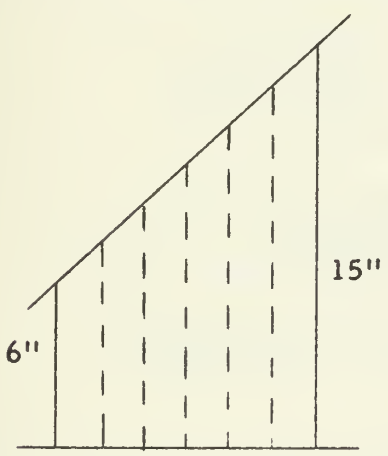
3. By induction, prove Theorem 142. [Hint. Part (i) of the proof is, of course, very easy. For part (ii), you will find Theorems 136 and 137 useful. Here is an analogous argument which may help you in formulating part (ii):

(1)	$a_1 + a_2 + a_3 = a_3 + a_2 + a_1$	["inductive hypothesis"]
(2)	$a_1 + a_2 + a_3 + a_4 = (a_1 + a_2 + a_3) + a_4$	["recursive definition"]
(3)	$= (a_3 + a_2 + a_1) + a_4$	[(1), (2)]
(4)	$= a_4 + (a_3 + a_2 + a_1)$	[cpa]
(5)	$= a_4 + a_3 + a_2 + a_1$	["Theorem 136"]

In the actual proof of the theorem, the application of Theorem 136 leaves one with an expression which still needs to be transformed by using Theorem 137.]

* * *

Consider the following problem:



A boy is building an incline 6" high on one end and 15" high on the other. He wishes to give it additional support by adding five more posts spacing them equally between the outer ones. How long should each additional post be?

It is easy to see that this is a problem in finding the terms of an arithmetic progression between the first term 6 and the seventh term 15. This problem is often stated as:

Insert 5 <u>arithmetic means</u> between 6 and 15.

Solve the problem.

* * *

K. Insert arithmetic means as indicated.

1. two between 4 and 10
2. three between 1 and 7
3. five between 8.4 and 3.6
4. nine between −6 and −14
5. two between 1.5 and −3
6. one between 8 and 12
7. one between x and y
8. three between x and y
9. n between x and y

* * *

The words 'arithmetic mean' are sometimes applied in a way different from that in Part K.

The <u>arithmetic mean of a sequence a</u> is the sequence $\bar{a}$ such that

$$\forall_n \; (\bar{a})_n = \frac{1}{n} \sum_{p=1}^{n} a_p .$$

So, for example, here are the first six terms of a sequence and the first six terms of its arithmetic mean:

a: 2, 8, 17, 21, 22, 30, ...

$\bar{a}$: 2, 5, 9, 12, 14, $\frac{50}{3}$, ... [Check this.]

Another example:

If a is the sequence such that, for each n, $a_n = n^2$,
then $\bar{a}$ is the sequence such that, for each n,

$$(\bar{a})_n = \frac{1}{n} \sum_{p=1}^{n} p^2 = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

The nth term of the arithmetic mean of a sequence is usually called <u>the arithmetic mean of the first n terms of the sequence</u>, or more commonly, <u>the average of the first n terms of the sequence</u>. You may have encountered problems in which you were asked to compute the arithmetic mean [or: the average] of certain numbers such as test scores or measurements. In such cases, you simply regard the numbers as the initial terms of a sequence and apply the definition. So, for example, if your

weekly test grades over a six-week period are

$$90, \quad 85, \quad 98, \quad 90, \quad 85, \quad \text{and} \quad 98,$$

then the arithmetic mean of these six grades is

$$\frac{1}{6}(90 + 85 + 98 + 90 + 85 + 98), \quad \text{or} \quad 91.$$

* * *

<u>L</u>. 1. (a) Find the arithmetic mean of the first twelve positive integers.

(b) Find the average of the first n odd positive integers.

(c) Find the arithmetic mean of the first n even positive integers.

(d) Prove that, for each n, the arithmetic mean of the first n terms of an AP is the average of its first and nth terms.

2. Prove that

(a) the arithmetic mean of a constant sequence is the sequence itself;

(b) the arithmetic mean of the sum of two sequences is the sum of their arithmetic means; and that

(c) the arithmetic mean of the product of a sequence by a constant function is the product of the arithmetic mean of the sequence by this constant function.

3. Bill Franklin's test scores for the first semester consisted of three 100s, four 95s, eight 90s, and one 85. What is the arithmetic mean of these scores?

☆<u>M</u>. Suppose that a is a sequence. Then, for each x, $a_p - x$ is said to be the <u>deviation</u> of the pth term from x.

1. Suppose that the first 10 terms of a sequence are 3, 8, 12, 10, 5, 17, 9, 15, 7, and 19.

(a) Compute the arithmetic mean of the first ten terms.

(b) Compute the sum of the deviations of the first ten terms from their arithmetic mean.

2. Exercise 1(b) suggests a theorem. State and prove it.

$$\forall_x \forall_n \left[\sum_{p=1}^{n} (a_p - x) = \qquad \Longleftrightarrow x = \qquad \right]$$

3. Consider the function f such that, for each x,

$$f(x) = \sum_{p=1}^{n} (a_p - x)^2 .$$

This is a quadratic function. Find the argument for which f has its minimum value.

INEQUATIONS

One of the theorems you proved in Unit 7 [Theorem 91] was like:

$$(*) \qquad \forall_x \forall_y \forall_u \forall_v\, [(x < y \text{ and } u < v) \Rightarrow x + u < y + v],$$

but with '>' instead of '<'. Obviously, (*) is also a theorem, and it suggests a theorem on continued sums. In fact, what (*) says is that, for any sequences a and b,

$$\forall_{m \leq 2}\ a_m < b_m \Rightarrow \sum_{p=1}^{2} a_p < \sum_{p=1}^{2} b_p .$$

More generally, we have the theorem:

Theorem 143.

$$\forall_n \left[\forall_{m \leq n}\ a_m < b_m \Rightarrow \sum_{p=1}^{n} a_p < \sum_{p=1}^{n} b_p \right]$$

Example. Prove that the sum of the first 50 terms of the sequence $\frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \ldots, \frac{1}{(p+1)^2+1}, \ldots$ is less than the sum of the first 50 terms of the sequence $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots, \frac{1}{2(p+1)}, \ldots$.

Solution. What we need to show is that

$$\sum_{p=1}^{50} \frac{1}{(p+1)^2+1} < \sum_{p=1}^{50} \frac{1}{2(p+1)}.$$

By Theorem 143, this result will follow if we can show that

$$\forall_{m \leq 50} \frac{1}{(m+1)^2+1} < \frac{1}{2(m+1)}.$$

Now, since $(m+1)^2+1 > 0$ and $2(m+1) > 0$,

$$\frac{1}{(m+1)^2+1} < \frac{1}{2(m+1)} \Longleftrightarrow (m+1)^2+1 > 2(m+1)$$

$$\Longleftrightarrow m^2+2m+2 > 2m+2.$$

But, since $m^2 > 0$, this is the case. So,

$$\forall_{m \leq 50} \frac{1}{(m+1)^2+1} < \frac{1}{2(m+1)}.$$

*

Part (i) of an inductive proof of Theorem 143 is trivial. Part (ii) follows easily from (*) on page 8-76.

For, suppose [inductive hypothesis] that

$$\forall_{m \leq q}\ a_m < b_m \Rightarrow \sum_{p=1}^{q} a_p < \sum_{p=1}^{q} b_p.$$

And, suppose that

$$\forall_{m \leq q+1}\ a_m < b_m.$$

From this latter assumption it follows that

$$\forall_{m \leq q}\ a_m < b_m \text{ and } a_{q+1} < b_{q+1}.$$

So, by the inductive hypothesis,

$$\sum_{p=1}^{q} a_p < \sum_{p=1}^{q} b_p \text{ and } a_{q+1} < b_{q+1}.$$

Hence, by (*),

$$\sum_{p=1}^{q} a_p + a_{q+1} < \sum_{p=1}^{q} b_p + b_{q+1}$$

and, by the recursive definition of Σ-notation, it follows that

$$\sum_{p=1}^{q+1} a_p < \sum_{p=1}^{q+1} b_p .$$

Hence,

$$\forall_{m \le q+1}\, a_m < b_m \Longrightarrow \sum_{p=1}^{q+1} a_p < \sum_{p=1}^{q+1} b_p .$$

Consequently,

$$\forall_n \left[\left(\forall_{m \le n}\, a_m < b_m \Longrightarrow \sum_{p=1}^{n} a_p < \sum_{p=1}^{n} b_p\right) \Longrightarrow \left(\forall_{m \le n+1}\, a_m < b_m \Longrightarrow \sum_{p=1}^{n+1} a_p < \sum_{p=1}^{n+1} b_p\right)\right].$$

Obviously, similar theorems hold in which '<' is replaced by '>', by '≥', and by '≤'. We shall refer to any of these as 'Theorem 143'.

EXERCISES

A. 1. Prove that the sum of the first n terms of the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2p}, \ldots$$

is greater than the sum of the first n terms of the sequence

$$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, \frac{1}{2p+1}, \ldots .$$

2. Prove: $\forall_n \sum_{p=1}^{n} \frac{4p-3}{2p-1} < \sum_{p=1}^{n} \frac{4p-1}{2p}$

3. Find the smallest m such that

$$\forall_{n \ge m} \sum_{p=m}^{n} (1 + 4p) > \sum_{p=m}^{n} (9 + 2p).$$

B. 1. Prove:

$$\forall_{x \geq 1} \ \frac{1}{2\sqrt{x}} < \sqrt{x} - \sqrt{x-1}$$

[Hint. Note that [for $a \geq 1$] $\frac{1}{2\sqrt{a}} < \frac{1}{\sqrt{a} + \sqrt{a-1}}$. Also, see the hint in Exercise 2 of Part I on page 8-71.]

2. Prove:

$$\forall_{x > 0} \ \frac{1}{2\sqrt{x}} > \sqrt{x+1} - \sqrt{x}$$

3. Use the theorems proved in Exercises 1 and 2, together with earlier theorems, to guess and prove a theorem of the form:

$$\forall_m \forall_{n \geq m} \quad \_\_ - \_\_ < \sum_{p=m}^{n} \frac{1}{2\sqrt{p}} < \_\_ - \_\_$$

4. In considering Problem IV on page 7-1 of Unit 7 you may have guessed that

$$\forall_{n > 1} \ 2\sqrt{n} - 2 < \sum_{p=1}^{n} \frac{1}{\sqrt{p}} < 2\sqrt{n} - 1.$$

Use the theorem of Exercise 3 to justify this guess. [Hint. For $m = 1$, the left inequation of Exercise 3 is '$\sqrt{n+1} - \sqrt{1} < \sum_{p=1}^{n} \frac{1}{2\sqrt{p}}$'. By Theorems 90 and 98b, $\forall_n \sqrt{n} < \sqrt{n+1}$. Also [if $n > 1$], for $m = 2$, the right inequation of Exercise 3 is '$\sum_{p=2}^{n} \frac{1}{2\sqrt{p}} < \sqrt{n} - \sqrt{1}$'. What does this tell you about $\sum_{p=1}^{n} \frac{1}{2\sqrt{p}}$?]

5. Complete.

(a) $\left[\!\left[\sum_{p=1}^{10^6} \frac{1}{\sqrt{p}} \right]\!\right] =$ (b) $\forall_{n > 1} \left[\!\left[\sum_{p=1}^{n^2} \frac{1}{\sqrt{p}} \right]\!\right] =$

☆6. Complete.

(a) $\left[\!\left[50 \sum_{p=10^4}^{10^6} \frac{1}{\sqrt{p}} \right]\!\right] =$

(b) $\forall_m \forall_{n > m} \forall_{q \leq m} \left[\!\left[q \sum_{p=m^2}^{n^2} \frac{1}{\sqrt{p}} \right]\!\right] =$

C. 1. A boy has ten blue socks and ten brown socks loose in a drawer. If he reaches into the drawer [without looking] and pulls out socks, one at a time, how many might he have to pull out to find two that matched?

2. A fruit-grower packs 500 boxes of apples. Each box contains at most 240 apples [and none is empty]. Show that at least three boxes contain the same number of apples.

3. Suppose, in Exercise 2, that each box contains more than 230 apples [and at most 240]. How many boxes, at least, must contain the same number of apples?

* * *

In answer to Exercise 1 of Part C, people sometimes say 'eleven'. [When they do this, what question are they answering?]

Exercises 1 and 2 probably look rather different. But, actually, each depends on the same idea. Let's analyze them to see what this idea is.

Exercise 1. If the boy pulls out a_1 blue socks and a_2 brown socks then he has a matching pair if either $a_1 > 1$ or $a_2 > 1$. Suppose that he doesn't. Then, $a_1 \leq 1$ and $a_2 \leq 1$, and the total number, $a_1 + a_2$, of socks he has pulled out is at most 2. So, if he pulls out more than 2 socks then either $a_1 > 1$ or $a_2 > 1$ [and he has a matching pair].

More formally,

$$\forall_{m \leq 2}\ a_m \leq 1 \Rightarrow \sum_{p=1}^{2} a_p \leq 2.$$

Consequently,

$$(*) \qquad \sum_{p=1}^{2} a_p > 2 \Longrightarrow \exists_{m \le 2}\ a_m > 1.$$

[Since the numbers a_p are integers, we can conclude that

$$\sum_{p=1}^{2} a_p \ge 3 \Longrightarrow \exists_{m \le 2}\ a_m \ge 2. \qquad \text{[Explain.]]}$$

Exercise 2. Suppose that, for each $m \le 240$, a_m is the number of boxes which contain m apples apiece. If there are not 3 boxes which contain the same number of apples then, for each $m \le 240$, $a_m \le 2$. In this case the total number of boxes is at most $2 \cdot 240$. So, if there are more than 480 boxes then there is some $m \le 240$ such that $a_m > 2$. Since $500 > 480$, there is some $m \le 240$ such that more than 2 boxes contain exactly m apples apiece.

More formally,

$$\forall_{m \le 240}\ a_m \le 2 \Longrightarrow \sum_{p=1}^{240} a_p \le 2 \cdot 240.$$

Consequently,

$$(**) \qquad \sum_{p=1}^{240} a_p > 2 \cdot 240 \Longrightarrow \exists_{m \le 240}\ a_m > 2.$$

Statements (*) and (**) suggest the following corollary of Theorem 143:

$$\forall_x \forall_n \left[\sum_{p=1}^{n} a_p > xn \Longrightarrow \exists_{m \le n}\ a_m > x \right]$$

A slightly more convenient form is:

Theorem 144.

$$\forall_x \forall_n \left[\sum_{p=1}^{n} a_p \ge x \Longrightarrow \exists_{m \le n}\ a_m \ge \frac{x}{n} \right]$$

Let's see how Theorem 144 is used in solving problems.

Problem 1. Suppose that fifty people attend a lecture. At least how many of these have birthdays in the same month?

Solution. We suppose that, for each $p \leq 12$, a_p is the number of people at the lecture who have birthdays during the pth month. Then,

$$\sum_{p=1}^{12} a_p = 50.$$

So, it follows from Theorem 144 that

$$\exists_{m \leq 12} \; a_m \geq \frac{50}{12}.$$

Since the numbers a_p are integers,

$$\exists_{m \leq 12} \, a_m \geq 5;$$

so there must be a month during which at least 5 of those present have their birthdays.

Problem 2. How many people must you have at a lecture to be sure that at least three of them have birthdays in the same month?

Solution. In this problem, we wish to know how many people must be present [that is, how large x must be] to justify the conclusion:

$$\exists_{m \leq 12} \; a_m > 2$$

Theorem 144 tells us that if $\frac{x}{12} > 2$ then this conclusion follows from:

$$\sum_{p=1}^{12} a_p \geq x$$

Since the least integer x such that $\frac{x}{12} > 2$ is 25, it follows that if there are as many as 25 people at the party, there are at least 3 who have birthdays in the same month.

* * *

D. 1. Use Theorem 144 to solve Exercises 2 and 3 of Part C.

2. How many people must be present in your class to insure that either at least three of them have birthdays in January or at least three of them do not have birthdays in January?

3. A pharmacist has five brown bottles of pills and three empty white bottles. What must be the total number of pills in the brown bottles in order to be sure that at least one of them contains enough pills so that when these pills are distributed among the white bottles, at least one of them will contain not less than ten pills?

4. Here are the first several terms of two sequences of positive integers.

(1) 9, 11, 16, 5, 60, ...
(2) 82, 76, 35, 30, 51, ...

If we divide each term of sequence (1) by 3, and record the remainders, it is not until we get to the fourth term that we find a remainder which we have found for an earlier term. For sequence (2), we find the same remainder a second time when we get to the second term.

(a) Given a sequence of positive integers, how far [that is, to what term] in the sequence might you have to go to find two terms which have the same remainder upon dividing by 3? By 4? By 5? By 6?

(b) Repeat (a) for two terms which have the same remainder upon dividing by 3, as well as upon dividing by 4. [For example, 17 and 41 are two such positive integers, but 17 and 37 are not.] By 3 and by 6? By 4 and by 6?

(c) How far in a sequence of positive integers might you have to go to find two terms which have the same remainder upon dividing by 3 and two terms which have the same remainder upon dividing by 4?

☆(d) How far in a sequence of positive integers might you have to go to find two terms which have the same remainder upon dividing by 3 as the remainder obtained by dividing by 4? [For example, 14 and 50 are two such positive integers, but 17 and 41 are not.]

* * *

In solving Exercise 4 of Part D you probably used a consequence of Theorems 144 and 106:

$$(\forall_m\ a_m \in I^+ \text{ and } \forall_n \sum_{p=1}^{n} a_p \geq n+1) \Rightarrow \exists_{m \leq n}\ a_m \geq 2$$

This theorem is sometimes called the pigeon-hole principle. It tells you that if you distribute n + 1 or more things among n boxes then at least one box will contain at least two things. So, for example, since the possible remainders upon dividing a positive integer by n are 0, 1, 2, ..., n - 1, it follows that if you divide each of as many as n + 1 positive integers by n then you are bound to get the same remainder at least twice.

* * *

☆E. 1. Show that if S is a set of n positive integers then there is a subset of S the sum of whose members is divisible by n. [Hint. Let a be a sequence whose first n terms are the members of S. How many different remainders may you get upon dividing the numbers $\sum_{p=1}^{q} a_p$, for $1 \leq q \leq n$, by n?]

2. Show that, given n + 1 positive integers, each less than or equal to 2n, there is at least one of them which is a factor of another. [Hint. Each positive integer has a greatest odd factor--for example, $12 = 4 \cdot 3$ and $8 = 8 \cdot 1$. It would be sufficient to show that two of the given numbers have the same greatest odd factor [Explain.]]

F. Each letter received at Zabranchburg High School is addressed either to the principal, Mr. Jones, or to one of the teachers. One day Mr. Jones' secretary told him that the mailman had delivered 49 letters.

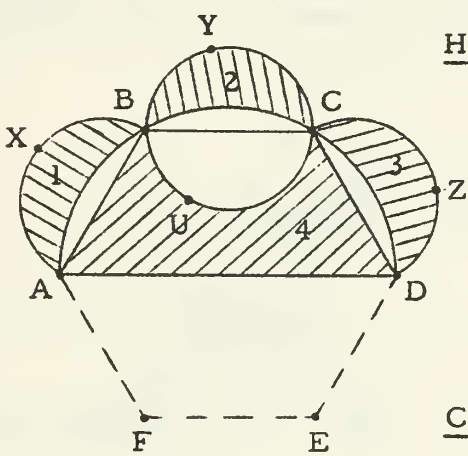
1. From his secretary's statement and his knowledge of the number of teachers, Mr. Jones concluded that someone received at least three letters. What can you conclude about the number of teachers at Z.H.S.?

2. From the information referred to in Exercise 1, Mr. Jones was not able to tell whether anyone had received more than three letters. What does this tell you about the number of teachers at Z.H.S.?

3. After reading his six letters, Mr. Jones learned that two teachers had received no mail that day. He was able to tell that some teacher had received at least four letters. How many teachers are there at Z.H.S.?

MISCELLANEOUS EXERCISES

1.



Hypothesis: ABCDEF is a regular hexagon, $\overset{\frown}{ABD}$, $\overset{\frown}{BUC}$, $\overset{\frown}{AXB}$, $\overset{\frown}{BYC}$, and $\overset{\frown}{CZD}$ are semicircles

Conclusion: $K_1 + K_2 + K_3 = K_4$

2. If a 26" long sample of cloth shrinks to 24.5" after washing, how many yards of this cloth should be used if you want 40 yards of it after washing?

3. Expand.

(a) $(2k^2 + k^3 - 7k^4 + 5)(k + 8k^2)$

(b) $(7x^4 - 6x^3y + 9x^2y^2 + 3xy^3 - y^4)(3x^2 - xy - y^2)$

4. Suppose that you were to graph the equations '$y = 2x + 2$' and '$(x - 2)^2 + (y - 5)^2 = 25$' on the number plane lattice [integer coordinates]. Would the graphs have any points in common?

5. If c cows consume f pounds of feed daily, how many pounds of feed are required each week for w cows?

6. A, B, and C are the vertices of a triangle, and AB = 7, BC = 8, and CA = 9. If P is a point of $\overleftrightarrow{AB}$ such that P belongs to the inscribed circle, what is AP?

7. Solve the equation '$\frac{5a + 4b}{3} = 7$' for 'b'.

8. If you pay \$80 for an article listed at \$90, what discount rate are you receiving?

9. Working together, A, B, and C can do a job in 2 days. But, just A and B can do it in 6 days, and just A and C can do it in 8 days. In how many days can each man do it alone?

10. Prove: $\forall_n \sum_{p=1}^{n} (2p - 1)^4 = \frac{n(4n^2 - 1)(12n^2 - 7)}{15}$

11. Simplify.

(a) $\frac{6x^4 - 13x^3 + 18x^2 - 23x + 10}{3x - 2}$ (b) $\frac{6x^4 - x^3 + 2x^2 - 2x - 1}{3x + 1}$

12. Solve these equations.

(a) $3x - 4 - \frac{4(7x - 9)}{15} = \frac{4}{5}\left(6 + \frac{x - 1}{3}\right)$ (b) $\frac{(3x - 2)(x - 1)}{21} = \frac{9}{7} + \frac{(x - 3)^2}{7}$

13. Suppose that S is the set of positive integers from 1 to n. The cartesian square $S \times S$ contains n^2 ordered pairs. Prove that the sum of the products obtained by multiplying the components for each pair is $\frac{n^2(n + 1)^2}{4}$.

14. Prove: $\forall_{x > 3}\ 1 < \frac{x + 1}{x - 1} < 2$

15. How many miles can a person walk in 55 minutes if he walks w miles in k hours?

16. Simplify.

(a) $(2a^3)^2$ (b) $(-ab^2)^2$ (c) $(-9p^3q^6)^2$ (d) $\left(-\frac{1}{3x^3}\right)^2$

(e) $(x^3y^2)^3$ (f) $(2x^2z^3)^3$ (g) $(-3p^4q^2)^3$ (h) $\left(-\frac{6}{5a^5b^2}\right)^3$

17. Complete:

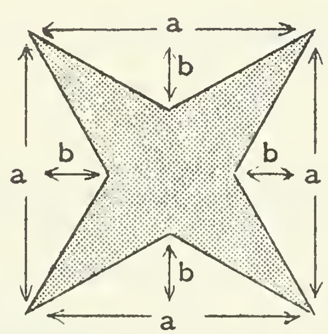
(a) $\forall_{a \neq 1}\ \frac{1}{1 - a} = 1 + \frac{\quad}{1 - a}$ (b) $\forall_{a \neq 1}\ \frac{1}{1 - a} = 1 + a + \frac{\quad}{1 - a}$

(c) $\forall_{a \neq 1}\ \frac{1}{1 - a} = 1 + a + a^2 + a^3 +$

18. Solve for 'x'.

(a) $\frac{ax}{bx - c} = \frac{d}{e}$ (b) $\frac{a}{px - q} = \frac{b}{p - qx}$ (c) $a(px + q) - b(rx - s) = c$

19.



Derive a formula for the area-measure K in terms of a and b.

20. Expand.

(a) $(x - 7)(x + 6)$ (b) $(y - 4)(y + 5)$ (c) $(c + 7)(c + 6)$

(d) $(2y - 1)(2y + 3)$ (e) $(4 - 3c)(4c + 3)$ (f) $(ac - 1)(ac + 3)$

21. Suppose that a and b are sequences such that, for each p, $a_p = 5 + 8p$ and $b_p = (p + 3)(p - 9)$. The tenth term of a is what term of b?

22. Prove: $\forall_k \forall_n \sum_{p=1}^{n} p(p + 2k - 1) = \frac{n(n + 1)(n + 3k - 1)}{3}$

23. Use the theorem '$\forall_x \forall_y\, x^2 - y^2 = (x + y)(x - y)$' in computing the square of 9999 in an easy way.

24. Solve these equations.

(a) $8x = 3(x + 20)$ (b) $7(x + 2) = 5(x + 8)$ (c) $4(n + 3.4) = 3(14.1 - n)$

25. A boy has a pile of pennies which he is trying to arrange in a solid square. The first time he tries, he has 116 left over. So he increases each side of the square by 3, and then finds that he is 25 coins short of completing a square. Can he succeed?

26. Simplify.

(a) $-xy + yz + zx + (-3xy - 2yz + 3zx) - (xy + yz - zx)$

(b) $pq - 3rp + 4qr + (3qr - pq) - (2pq - 3qr + 4rp)$

(c) $a - 2b + 3c - d - (b - 3c) - (2c + 3d) + (2a - b + d)$

27. If $4(2A - B) = 3(2B + A)$, find the ratio of A to B.

28. True-or-false?

$\{(x, y): 3x + y = 2 \text{ and } 7x + 4y = 3\} \cap \{(x, y): 2x + 5y = -3 \text{ and } 5x + 2y = 3\} = \emptyset$

29. Simplify.

(a) $\frac{2x^4 + 7x^3 - 7x^2 - 20x + 12}{x + 2}$ (b) $\frac{10x^4 - 11x^3 - 5x^2 + 5x - 2}{2x^2 - x - 2}$

30. (a) Expand: $(x - y)^2 + (y - z)^2 + (z - x)^2$

(b) Prove: $\forall_x \forall_y \forall_z\, x^2 + y^2 + z^2 - xy - yz - zx \geq 0$

31. Solve for 'x'.

(a) $\frac{x - a}{p} + \frac{x - b}{q} = \frac{x - c}{r}$ (b) $a(px - q) = b(px - q) + c$

32. The sum of an integer and its square is 6 times the next integer. Find this integer.

33. Consider the sequence a such that, for each p, $a_p = p^2 - 3p + 43$.

(a) Is there a term of this sequence which is not a prime number?

(b) If your answer to (a) is 'yes', what is the first term of the sequence which is a composite number?

34. Expand.

(a) $(7z^2 + 3z + 4)(8z^4 + 3z + 7)$ (b) $(2x^7 + 5x^5 + 6x^2 + 3)(2x^3 + 5x + 1)$

35. Suppose that $A = 1 - \frac{1}{B}$ and $B = 1 + \frac{2}{C}$. Show that $A + B + C = \frac{B^3 + 1}{B(B - 1)}$.

36. At a certain meeting, each person present shakes hands just once with everyone else. Altogether, 1225 handshakes were exchanged. How many people were present?

37. Solve the system of equations:

$$\begin{cases} 3x - 2y + 4z = -10 \\ 5x - 2y + 6z = -13 \\ 6x + 4y - 2z = 5 \end{cases}$$

38. Suppose that a tank can be filled by two inlet pipes, one at the rate of i_1 gallons per minute and the other at the rate of i_2 gallons per minute, and that the tank can be emptied by an outlet pipe at the rate of e gallons per minute. Assuming that there were V_0 gallons in the tank to begin with, write a formula for the number of gallons (V) of water in the tank after all three pipes have been operating for t minutes.

39. (a) Prove by induction: $\forall_n \frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n \in I^+$

(b) Give another proof based on the algebra theorem:

$$\forall_x\ 3x^5 + 5x^3 + 7x = 3(x - 2)(x - 1)x(x + 1)(x + 2) + 4 \cdot 5(x - 1)x(x + 1) + 15x$$

40. Find two consecutive odd numbers whose product is 255.

41. If a root of the equation '$x^3 + 5x^2 - 6x - 1440 = 0$' is 10, what is a root of the equation '$(2x-5)^3 + 5(2x-5)^2 - 6(2x-5) - 1440 = 0$'?

42. Reduce.

(a) $\dfrac{cy + dy}{ay - by}$ (b) $\dfrac{3ma + pa}{3mb + pb}$ (c) $\dfrac{4p^2 - 20pq + 25q^2}{12p^2 - 16pq - 35q^2}$

43. Suppose that f is a function such that, for some numbers a and b, $f(x) = 1 + \dfrac{a}{1+x} + \dfrac{b}{1+x^2}$. If $(1, 5) \in f$ and $(2, 3) \in f$, find $f(3)$.

44. Bill opens a book at a certain page and chooses one of the first 9 words in one of the first 9 lines. He multiplies the page number by 10, adds 25, and then adds the line number. He multiplies this last sum by 10, and adds the word number. The result is 12697. Give the page number, the line number, and the word number.

45. The incomes of Mr. French and Mr. Muller are in the ratio of 3 to 2 but their expenditures are in the ratio of 5 to 3. If each saves $1000 a year, what are their incomes?

46. Suppose that a and b are sequences such that $a_n = 2n - 1$ and $b_n = 95 - 6n$. Find the number p such that $a_p = b_p$.

47. Simplify.

(a) $[6(x+y) - 3(x-y)] + [9(x-y) - 7(x+y)] - [7(x+y) - 3(x-y)]$

(b) $[\frac{4}{5}(m-3n) + \frac{3}{4}(m+n)] - [-(m-3n) - \frac{1}{2}(m+n)] + [\frac{1}{5}(m-3n) + \frac{1}{4}(m+n)]$

48. Show that if $\dfrac{X}{B-C} = \dfrac{Y}{C-A} = \dfrac{Z}{A-B}$ then $X + Y + Z = 0$.

49. Solve these equations.

(a) $\frac{2}{3}(x+7) = \frac{3}{5}(2x-3)$ (b) $\dfrac{x}{2x-7} = \dfrac{5}{3}$ (c) $\dfrac{4x-7}{7x+4} = \dfrac{5}{12}$

50. Find two numbers which differ by 39 and whose ratio is 4 : 7.

51. Water is turned on into an empty tank at a constant rate. After several minutes the flow is increased to another steady rate. If the tank contained 6 gallons at the end of 2 minutes, 15 gallons at the end of 5 minutes, 31 gallons at the end of 9 minutes, and 41 gallons at the end of 11 minutes, when was the rate of flow changed?

52. Simplify.

(a) $\frac{x^2y^3}{x^2y}$ (b) $\frac{21x^3y^2}{3xy}$ (c) $\frac{8b^3c^5}{-4c}$ (d) $\frac{63x^4y^2z^7}{9x^4y^2}$

EXPLORATION EXERCISES

A. In each exercise you are given a recursive definition of a sequence. List, in unsimplified form, the first four terms and the ninth term of the sequence.

Sample. $f_1 = 3$

$\forall_n \; f_{n+1} = f_n \cdot (n + 2)$

Solution.

$3,\; 3(1 + 2),\; 3(1 + 2)(2 + 2),\; 3(1 + 2)(2 + 2)(3 + 2);$

$3(1 + 2)(2 + 2)(3 + 2)(4 + 2)(5 + 2)(6 + 2)(7 + 2)(8 + 2)$

1. $f_1 = 5$

$\forall_n \; f_{n+1} = f_n \cdot n$

2. $f_1 = 1$

$\forall_n \; f_{n+1} = f_n \cdot (n + 1)$

3. $f_1 = 3$

$\forall_n \; f_{n+1} = f_n \cdot 3$

4. $f_1 = \frac{1}{2}$

$\forall_n \; f_{n+1} = f_n \cdot \left(\frac{n + 1}{n + 2}\right)$

5. $f_1 = 1$

$\forall_n \; f_{n+1} = f_n\sqrt{\frac{n + 1}{n}}$

6. $f_1 = 3$

$\forall_n \; f_{n+1} = f_n \cdot \left(1 + \frac{2}{n + 1}\right)$

B. Write simple expressions for the 100th terms of the sequences in Exercises 3, 4, 5, and 6 of Part A.

8.02 Continued products.--In analogy with continued sums, we can also define continued products--that is, products of successive terms of a given sequence. Just as 'Σ' is the customary notation for continued sums, so the Greek letter 'Π' is used in referring to continued products. We begin with the recursive definition for the continued product sequence of a sequence a.

$$(*)\qquad \begin{cases} \prod_{p=1}^{1} a_p = a_1 \\ \forall_n \prod_{p=1}^{n+1} a_p = \prod_{p=1}^{n} a_p \cdot a_{n+1} \end{cases}$$

EXERCISES

A. Expand.

1. $\prod_{p=1}^{4} p$ 2. $\prod_{p=1}^{1} (3p-1)^2$ 3. $\prod_{p=1}^{6} 3$ 4. $\prod_{q=1}^{8} (1 + \frac{1}{q})$

B. Prove by mathematical induction.

1. $\forall_n \prod_{p=1}^{n} (1 + \frac{1}{p}) = n + 1$ 2. $\forall_n \prod_{p=1}^{n} (1 + \frac{2p+1}{p^2}) = (n+1)^2$

EXTENDING Π-NOTATION

In using Σ-notation we found it desirable to extend our original recursive definition in two ways. First, we adopted the convention:

$$\sum_{p=1}^{0} a_p = 0,$$

which was suggested by the principle for adding 0. Then, we made a

modification to take care of functions whose domains are any of the sets $\{k: k \geq j\}$, for $j \in I$, and also to take care of continued sums of successive terms of a sequence, starting at any term.

It is useful to define Π-notation in similar generality. To do so, we must first decide what will be the most convenient meaning for:

$$\prod_{p=1}^{0} a_p$$

It would certainly be inconvenient if we chose a definition of Π-notation which was not consistent with (*). In fact, the two sentences of (*) should, as in the case of Σ-notation, be consequences of our new definition. Our experience with Σ-notation suggests that what we want is a definition of the form:

$$\begin{cases} \displaystyle\prod_{p=1}^{0} a_p = \\ \forall_{k \geq 0} \displaystyle\prod_{p=1}^{k+1} a_p = \prod_{p=1}^{k} a_p \cdot a_{k+1} \end{cases}$$

From the second of these sentences it follows that

$$\prod_{p=1}^{1} a_p = \prod_{p=1}^{0} a_p \cdot a_1 . \qquad \text{[Check this.]}$$

If the new definition is to be consistent with (*), we must define '$\prod_{p=1}^{0} a_p$' so that

$$a_1 = \prod_{p=1}^{0} a_p \cdot a_1 .$$

Except in the case of sequences whose first terms are 0, this leaves us no choice:

$$\prod_{p=1}^{0} a_p = 1 \qquad \text{[Why the exception?]}$$

And, since it would be inconvenient for $\prod_{p=1}^{0} a_p$ to have one value for sequences for which $a_1 \neq 0$ and to have another value [or to be undefined] for sequences for which $a_1 = 0$, we shall adopt the same convention for all sequences.

The final form of the recursive definition for Π-notation is now easy to give:

For each $j \in I$ and each function a whose domain includes $\{k: k \geq j\}$,

$$\begin{cases} \prod_{i=j}^{j-1} a_i = 1 \\ \forall_{k \geq j-1} \prod_{i=j}^{k+1} a_i = \prod_{i=j}^{k} a_i \cdot a_{k+1} \end{cases}$$

[From now on, to save words, we shall call any function whose domain is one of the sets $\{k: k \geq j\}$, $j \in I$, a sequence.]

Example. Use Π-notation to express the product of the first ten terms of the sequence a such that $a_k = k$, for all $k \geq -5$.

Solution. $\prod_{i=-5}^{4} i$ [What number is this?]

EXERCISES

A. Compute.

1. $\prod_{i=0}^{5} (2i + 1)$
2. $\prod_{i=-2}^{3} (2i + 1)$
3. $\prod_{k=0}^{3} 2$

4. $\prod_{i=-4}^{-1} 2$

5. $\prod_{i=7}^{6} (3i + 2)$

6. $\prod_{i=7}^{7} (3i + 2)$

7. $\prod_{p=1}^{0} p^2$

8. $\prod_{p=1}^{0} 5$

9. $\prod_{i=-3}^{-1} i \cdot \prod_{i=1}^{3} i$

B. For each equation, find a value of 'x' which satisfies it.

1. $\prod_{i=3}^{5} (5i + 2) = (5 \cdot 3 + 2)(5x + 2)(5 \cdot 5 + 2)$

2. $\sum_{i=3}^{5} (5i + 2) = (5 \cdot 3 + 2) + (5x + 2) + (5 \cdot 5 + 2)$

3. $\prod_{i=-2}^{0} (7i + 3) = (7 \cdot -2 + 3)(7 \cdot -1 + 3)x$

4. $\prod_{p=1}^{5} (3p) = 3^x \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

5. $\prod_{p=1}^{x} [3p(p - 2)] = 1$

6. $\prod_{i=-3}^{x} (i^2 + 3) = \prod_{i=-3}^{19} (i^2 + 3) \cdot (20^2 + 3)$

7. $\prod_{p=1}^{12} (p^2 - 3p + 1) = \prod_{p=1}^{11} (p^2 - 3p + 1) \cdot x$

8. $\prod_{p=7}^{12} (p^2 - 3p + 1) = \prod_{p=7}^{10} (p^2 - 3p + 1) \cdot x$

9. $\prod_{p=1}^{x}(2p + 3) = \prod_{p=1}^{7}(2p + 3)\cdot 399$

10. $\prod_{i=1}^{7}(i + 3) = \prod_{i=1}^{3}(i + 3)\cdot\prod_{i=x}^{7}(i + 3)$

11. $\prod_{p=1}^{18}(3p + 7) \div \prod_{p=x}^{18}(3p + 7) = 10$

12. $\prod_{p=1}^{20}(2p + 9) \div \prod_{p=2}^{19}(2p + 9) = x$

13. $\prod_{p=1}^{x} p = 720$

14. $\prod_{p=1}^{x} 2 = 128$

15. $\prod_{i=0}^{x} 2 = 128$

16. $\prod_{p=4}^{6}(2p + 5) = \prod_{p=1}^{x}[2(p + 3) + 5]$

17. $\prod_{i=-2}^{11}(5i - 1) = \prod_{i=1}^{x}[5(i - 3) - 1]$

18. $\prod_{p=3}^{7}(6p + 3) = \prod_{p=1}^{5}[6(p + x) + 3]$

19. $\prod_{p=1}^{6}(1 + \frac{1}{p})\cdot\prod_{p=1}^{6}(1 - \frac{1}{p + x}) = 1$

C. 1. Suppose that f is a sequence such that, for each n,

$$f_n = \prod_{p=1}^{n} \left(1 + \frac{p}{10}\right).$$

Graph the first 10 ordered pairs of f.

2. Suppose that f is a sequence such that, for each n,

$$f_n = \prod_{p=1}^{n} \frac{p}{p+1}.$$

Find the smallest m such that, for all $n > m$, $f_n < 0.01$.

3. Suppose that g is a sequence such that, for each $k \geq 0$,

$$g_k = \prod_{p=1}^{k} 2.$$

(a) Complete the following table.

k	0	1	2	3	4	5	6	7
g_k				8				
$\sum_{i=0}^{k} g_i$				15				

(b) Study the table and guess and prove a theorem which begins:

$$\forall_n \sum_{i=0}^{n-1} g_i = \qquad - 1$$

4. Suppose that g is a sequence such that, for each $k \geq 0$,

$$g_k = \prod_{p=1}^{k} p.$$

Find the smallest m such that, for all $n > m$, $g_n > 1000000$.

D. <u>The factorial sequence</u> is defined by:

$$\forall_{k \geq 0}\ k! = \prod_{p=1}^{k} p$$ [Read 'k!' as 'k factorial' or as 'factorial k'.]

Equivalently:

$$\begin{cases} 0! = 1 \\ \forall_n\ n! = (n-1)! \cdot n \end{cases}$$

1. Complete the following table.

k	0	1	2	3	4	5	6	7
k!								

2. Compute.

(a) $\frac{5!}{3!}$ (b) $\frac{12!}{11!}$ (c) $\frac{3!}{0!}$ (d) $\frac{9!}{13!}$

(e) $\frac{(8-2)!}{8!6!}$ (f) $\frac{14!}{8!6!}$ (g) $\frac{5!}{1!2!3!4!5!}$

(h) $\frac{(3+2)!}{(3-2)!}$ (i) $\frac{19!}{10!9!}$ (j) $\frac{9!}{6!3!} + \frac{9!}{5!4!}$

3. Use Theorem 138 to complete and prove each of the following.

(a) $\forall_n \sum_{p=1}^{n} p \cdot p! =$ [<u>Hint</u>. $\forall_p\ (p+1) \cdot p! = (p+1)!$]

(b) $\forall_n \sum_{p=1}^{n} \frac{p}{(p+1)!} =$

4. Prove these theorems.

(a) $\forall_n\ n! \geq n$ (b) $\forall_n \sum_{k=0}^{n} \frac{1}{k!} \leq 3 - \frac{1}{n!}$

* * *

Some of the theorems of section 8.01 concerning continued sums have obvious analogues for continued products. To prove them one need only take proofs of the Σ-theorems and replace 'Σ's by 'Π's, '+'s by '$\cdot$'s, and, in first steps of recursive definitions, '0's by '1's. Here are five such Π-theorems. Some of them you may have discovered in solving the exercises of Part B on pages 8-95 and 8-96. In each case, identify the corresponding Σ-theorem, and suggest a descriptive name:

<u>Theorem 145.</u>

$$\forall_j \forall_{k \geq j-1} \prod_{i=j}^{k} (a_i b_i) = \prod_{i=j}^{k} a_i \cdot \prod_{i=j}^{k} b_i$$

<u>Theorem 146.</u>

$$\forall_j \forall_{j_1 \geq j-1} \forall_{k \geq j_1} \prod_{i=j}^{k} a_i = \prod_{i=j}^{j_1} a_i \cdot \prod_{i=j_1+1}^{k} a_i$$

<u>Theorem 147.</u>

$$\forall_j \forall_{k \geq j} \prod_{i=j}^{k} a_i = a_j \cdot \prod_{i=j+1}^{k} a_i$$

<u>Theorem 148.</u>

$$\forall_j \forall_{j_1} \forall_{k \geq j-1} \prod_{i=j}^{k} a_i = \prod_{i=j+j_1}^{k+j_1} a_{i-j_1}$$

<u>Theorem 149.</u>

$$\forall_j \forall_{k \geq j-1} \prod_{i=j}^{k} a_i = \prod_{i=j}^{k} a_{k+j-i}$$

There are some striking omissions from the foregoing list of analogues of Σ-theorems. For example:

$$(\Sigma) \qquad \forall_x \forall_{k \geq 0} \sum_{p=1}^{k} x = xk$$

is a very useful Σ-theorem. Is there an analogous Π-theorem?

$$(\Pi) \qquad \forall_x \forall_{k \geq 0} \prod_{p=1}^{k} x = \ ?$$

The Σ-theorem (Σ) says that,

$$\forall_x \forall_{k \geq 0} \sum_{p=1}^{k} x \text{ is the kth multiple of } x.$$

We can complete the Π-theorem (Π) if we can complete:

$$\forall_x \forall_{k \geq 0} \prod_{p=1}^{k} x \text{ is the kth} \qquad\qquad \text{of } x.$$

Can you complete this sentence?

EXPONENTIAL SEQUENCES

Your knowledge of exponents probably enabled you to complete the theorem (Π). In fact, the generalization:

$$(*) \qquad \forall_x \forall_{k \geq 0} x^k = \prod_{p=1}^{k} x$$

defines, for each x, what we shall call <u>the exponential sequence with base x</u>. So, for example, the first five terms of the exponential sequence with base 2 are

$$2^0, \quad 2^1, \quad 2^2, \quad 2^3, \quad 2^4.$$

The definition $(*)$ tells us that these terms are

$$\prod_{p=1}^{0} 2, \quad \prod_{p=1}^{1} 2, \quad \prod_{p=1}^{2} 2, \quad \prod_{p=1}^{3} 2, \quad \prod_{p=1}^{4} 2.$$

And, the definition of Π-notation and our knowledge of multiplication tells us that these terms are

$$1, \quad 2, \quad 4, \quad 8, \quad 16.$$

Note that the (k + 1)th term of the exponential sequence with base x is the kth power of x.

Of course, exponential sequences can be defined without Π-notation. Here is such a definition:

$$\begin{cases} \forall_x \; x^0 = 1 \\ \forall_x \forall_{k \geq 0} \; x^{k+1} = x^k \cdot x \end{cases}$$

[Note. In some discussions of exponents the symbol '0^0' is left undefined. As we have indicated in the discussion of Π-notation on page 8-94, it is usually inconvenient to leave expressions undefined.]

EXERCISES

A. Compute.

1. $\prod_{p=1}^{0} -3$ 2. $\prod_{p=1}^{0} 8$ 3. $\prod_{p=1}^{0} 0$ 4. $\prod_{p=1}^{3} 1 + \prod_{p=1}^{2} 0$

5. 6^0 6. 1^8 7. 0^{15} 8. 0^0

9. 2^{10} 10. 2^{12} 11. 2^8 12. 2^{20}

13. -1^4 14. $(-1)^4$ 15. $(5-2)^5$ 16. $(2-5)^5$

17. $3^4 \cdot 3^5$ 18. $3^4 + 3^5$ 19. $2^{12} - 2^{12}$ 20. $3^{10} - 3^9$

B. 1. Use Theorem 145 and the definition of exponential sequences to prove:

$$\forall_x \forall_y \forall_{k \geq 0} \; (xy)^k = x^k y^k$$

2. Use Theorems 146 and 148 and the definition of exponential sequences to prove:

$$\forall_x \forall_{j \geq 0} \forall_{k \geq 0} \; x^{j+k} = x^j x^k$$

C. In each exercise, you are given a pair of equations, one in 'y' and 'k', the other in 'y' and 'x'. Graph each pair on the same chart, the first for $0 \le k \le 5$, and the second for $0 \le x \le 5$.

1. $y = 2^k$
   $y = x^2$
2. $y = 0^k$
   $y = x^0$
3. $y = 1^k$
   $y = x^1$

D. Graph each pair of equations on the same chart for $0 \le k \le 10$, using crosses for the first equation and heavy dots for the second.

1. $y = -1^k$
   $y = (-1)^k$
2. $y = (-1)^{2k}$
   $y = (-1)^{2k+1}$
3. $y = (-1)^k$
   $y = (-1)^{k+2}$

E. Graph these sets.

1. $\{(k, y), k \ge 0: y^k = 0\}$
2. $\{(k, y), k \ge 0: y^k > 0\}$
3. $\{(k, y), k \ge 0: y^{k+1} > y^k\}$
4. $\{(k, y), k \ge 0: y^k \le 0\}$

F. On the same chart, sketch the graphs of these equations for $0 \le x \le 3$:

$$y = x^0, \quad y = x^1, \quad y = x^2, \quad y = x^3$$

* * *

The work you did in the preceding exercises undoubtedly suggested many theorems about exponential sequences. You will be asked to prove some of these theorems in the next set of exercises. As a sample, here is a proof by induction of the theorem which is suggested by Exercise 3 of Part D.

Prove:

Theorem 150b.

$$\forall_{k \ge 0}\ (-1)^{k+2} = (-1)^k$$

(i) $(-1)^{0+2} = (-1)^2 = 1 = (-1)^0$

(ii) Suppose that $(-1)^{j+2} = (-1)^j$. Since $(-1)^{(j+1)+2} = (-1)^{j+2+1}$,

it follows, by the recursive definition, that

$$(-1)^{j+1+2} = (-1)^{j+2}(-1).$$

Hence,

$$(-1)^{j+1+2} = (-1)^{j}(-1).$$

But, again by the recursive definition,

$$(-1)^{j}(-1) = (-1)^{j+1}.$$

Consequently,

$$(-1)^{j+2} = (-1)^{j} \Longrightarrow (-1)^{j+1+2} = (-1)^{j+1}.$$

(iii) The theorem follows from (i) and (ii) by mathematical induction [actually, by Theorem 114].

* * *

G. Prove:

1. $\forall_{k \geq 0}\ 1^k = 1$ [Theorem 150a]

2. $\forall_{k \geq 0}\ [(-1)^{2k} = 1 \text{ and } (-1)^{2k+1} = -1]$ [Theorem 150c]

3. $0^0 = 1 \text{ and } \forall_n\ 0^n = 0$ [Theorem 150d]

4. $\forall_m \forall_{k \geq 0}\ m^k \in I^+$ [Theorem 151a]

5. $\forall_{x > 0} \forall_{k \geq 0}\ x^k > 0$ [Theorem 152a]

6. $\forall_{x \neq 0} \forall_{k \geq 0}\ x^k \neq 0$ [Theorem 152b]

7. $\forall_{x > 1} \forall_{k \geq 0}\ x^{k+1} > x^k$ [Theorem 152c]

H. 1. Use Theorem 138 and:

$$\forall_x \forall_{k \geq 0}\ x^{k+1} - x^k = (x-1)x^k$$

to prove:

$$\forall_x \forall_{k \geq 0}\ (x-1)\sum_{p=1}^{k} x^{p-1} = x^k - 1 \qquad \text{[Theorem 153]}$$

2. Compute.

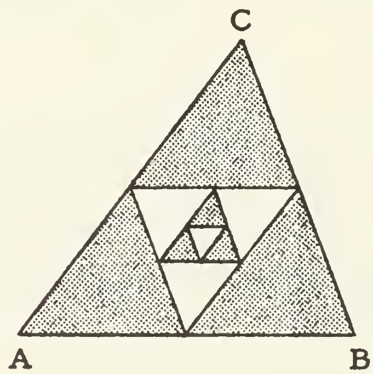
(a) $\sum_{p=1}^{8} 3^{p-1}$ (b) $\sum_{p=1}^{10} \left(\frac{1}{2}\right)^{p-1}$ (c) $\sum_{p=1}^{10} (-2)^{p-1}$

(d) $\sum_{p=5}^{10} (-1)^{p-1}$ (e) $\sum_{p=4}^{10} 2^{p-1}$ (f) $\sum_{p=1}^{100} 0^{p-1}$

(g) $\sum_{p=1}^{7} 5^{p}$ (h) $\sum_{p=1}^{10} 1^{p-1}$

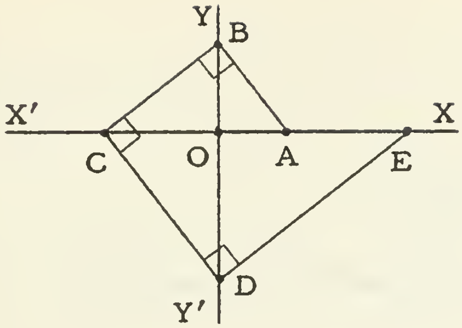
3. Solve Problem II on page 7-1 of Unit 7.

4. Consider a sequence of triangles constructed in the manner illustrated. The first triangle is ΔABC, the vertices of the second are the mid-points of the sides of the first, the vertices of the third are the mid-points of the sides of the second, and so on.



(a) If the perimeter of the tenth triangle is 1, what is the sum of the perimeters of the first twenty triangles?

(b) Certain regions of the 1st, 3rd, 5th, 7th, ... triangles are shaded as shown. If the area-measure of ΔABC is k, what is the sum of the area-measures of the shaded regions of the first ten triangles?

5. 

<u>Hypothesis</u>: $\overleftrightarrow{X'X} \perp \overleftrightarrow{YY'}$ at O,
$\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{CD}$,
$\overline{CD} \perp \overline{DE}$,
$OA = 1$, $OB = b$

<u>Conclusion</u>: $OC = b^2$, $OD = b^3$,
$OE = b^4$

6. One of the things you can do with Theorem 153 is to use it in factoring expressions of the form '$x^k - 1$'. For example,

a factorization of '$x^2 - 1$' is '$(x - 1)(x + 1)$',
a factorization of '$x^3 - 1$' is '$(x - 1)(x^2 + x + 1)$',
and a factorization of '$x^4 - 1$' is '$(x - 1)(x^3 + x^2 + x + 1)$'.

Give a factorization of each of the following:

(a) $x^7 - 1$ (b) $y^6 - 1$ (c) $1 - z^4$ (d) $8x^3 - 1$

7. Give one factor [with respect to I^+] of each of the listed numbers.

(a) $3^2 - 1$ (b) $4^3 - 1$ (c) $5^4 - 1$ (d) $10^9 - 1$ (e) $100^{99} - 1$

8. Exercise 7 suggests a theorem which is easily proved with the help of Theorem 153. Complete and prove:

$$\forall_n\ n \mid \qquad\quad - 1$$

9. Compute $2^1 - 1$, $3^2 - 1$, $4^3 - 1$, $5^4 - 1$, and $6^5 - 1$. Guess a stronger divisibility theorem than that of Exercise 8.

☆10. Prove the theorem you guessed in Exercise 9. [<u>Hint</u>. The proof in Exercise 8 proves that $\forall_m \forall_n\ n \mid (n + 1)^m - 1$. Also, with the help of Theorem 153 and various summation principles, you can show that

$$(n + 1)^n - 1 = n\left[\sum_{p=2}^{n} [(n + 1)^{p-1} - 1] + n\right].]$$

☆BASE-m REPRESENTATION OF POSITIVE INTEGERS

From Part ☆J beginning on page 7-107 of Unit 7, and from the ensuing discussion, you guessed that, for each integer $m > 1$, each positive integer n has a base-m representation. [You are very well acquainted with this when m is 10.] That is, there are integers n_k such that $0 \le n_k < m$ and such that, for a sufficiently large integer p,

$$n = \sum_{k=0}^{p-1} n_k \cdot m^k .$$

[In the case of the number, n, of states in the United States and $m = 10$, what is the sufficiently large p and what are the numbers n_k? (Answer these questions for $m = 6$.)] We are now in a position to prove that this is the case and to confirm the conjecture that suitable "digits" n_k can be computed from the formula:

$$(*) \qquad n_k = \left\{\frac{[\![n/m^k]\!]}{m}\right\} m$$

To begin with, Theorem 124 and the definition of the exponential sequence with base m have as a consequence:

$$\forall_n \forall_{k \ge 0} \forall_m \; 0 \le \left\{\frac{[\![n/m^k]\!]}{m}\right\} m = \left[\!\!\left[\frac{n}{m^k}\right]\!\!\right] - \left[\!\!\left[\frac{n}{m^{k+1}}\right]\!\!\right] m < m$$

In particular, the numbers n_k given by (*) are nonnegative integers [Why?] which are less than m. Moreover, for each $p \in I^+$,

$$\sum_{k=0}^{p-1} \left(\left\{\frac{[\![n/m^k]\!]}{m}\right\} m\right) m^k = \sum_{k=0}^{p-1} \left(\left[\!\!\left[\frac{n}{m^k}\right]\!\!\right] - \left[\!\!\left[\frac{n}{m^{k+1}}\right]\!\!\right] m\right) m^k .$$

From this it is easy to show that

$$\sum_{k=0}^{p-1} \left(\left\{\frac{[\![n/m^k]\!]}{m}\right\} m\right) m^k = n - \left[\!\!\left[\frac{n}{m^p}\right]\!\!\right] m^p .$$

Do so now.

To complete the proof that n has a base-m representation, with digits given by (*), all that remains is to show that there is a positive integer p such that

$$\left[\!\left[\frac{n}{m^p}\right]\!\right] m^p = 0.$$

And, to do this, all we need do is show that there is a positive integer p such that $m^p > n$. For, if this is the case then, since $n > 0$,

$$0 < \frac{n}{m^p} < 1,$$

and $\left[\!\left[\frac{n}{m^p}\right]\!\right] = 0$.

☆EXERCISES

A. 1. Prove: $\forall_{m > 1} \forall_{k \geq 0}\, m^k > k$ [Theorem 151b]

[Hint (for part (ii) of an inductive proof). By Exercise 7 of Part G on page 8-103, $m^{k+1} > m^k$. So, using Exercise 4 of Part G, $m^{k+1} \geq m^k + 1$.]

2. Let p be the least $k \geq 0$ such that $m^k > n$ [How do you know that there is such an integer p?]. Since $m^p > n \geq 1$, $p \neq 0$ [Why?]. So, $m^{p-1} \leq n < m^p$. Show that $n_{p-1} \neq 0$.

B. 1. Show that each positive integer n has a factorial representation --that is, show that there are integers n_k such that $0 \leq n_k < k + 2$ and such that, for a sufficiently large integer p,

$$n = \sum_{k=0}^{p-1} n_k \cdot (k+1)!$$

[Hint. Consider the instance of Theorem 124 for $x = n/k!$ and $m = k + 1$.]

2. In terms of 'n', what is one "sufficiently large p"?

3. Use an algorithm analogous to the one for base-m representations [see page 7-110 for the case m = 6] to find the digits in the factorial representation of 39. Of 75621. Of 40320.

MISCELLANEOUS EXERCISES

A. 1. Draw a square ABCD. Let E be the midpoint of $\overleftrightarrow{AD}$ and F be the point on $\overrightarrow{EA}$ such that EF = EB. Let H be the point of $\overline{AB}$ such that AH = AF. If G is the fourth vertex of the square whose other three vertices are F, A, and H, and if K is the point of intersection of $\overrightarrow{GH}$ and $\overline{DC}$, show that the area-measure of square AFGH is the area-measure of quadrilateral KHBC.

2. A dealer buys a certain article for \$16. At what price should he list the article so that if he gives a discount of 30%, he can still make a 20% profit on the selling price?

3. Simplify.

   (a) $100 \div 10 - 12 \cdot \frac{1}{2}$ (b) $\dfrac{20 - 14 \cdot \frac{1}{2}}{6 \cdot \frac{1}{3}}$ (c) $\dfrac{8}{x + 3} - \dfrac{2}{x}$

4. Find at least one substitution for 'k' such that the equation '$2x^2 + kx + 36 = 0$' has two roots, one of which is twice the other.

5. It takes Emma 30 minutes to type a certain report, but Consuela can type it in 20 minutes. How long would it take them working together?

6. Suppose that a is the measure of one of the congruent sides of an isosceles triangle and b is the measure of its base. Derive a formula for K, the area-measure of the triangle, in terms of a and b.

7. Write a quadratic equation whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

8. In Zabranchburg, 17 students graduate from high school out of every 20 who enter, and 7 out of every 10 who graduate go to college. What per cent of the students who enter high school finally enter college?

9. Solve the equation: $\frac{1}{8} - \frac{1}{x} = \frac{1}{12}$

10. Suppose that $A - C = 3$ and $B + C = 9$. Derive a formula for A in terms of B.

11. Solve the equation: $\frac{x}{x + 5} + \frac{1}{x - 1} = \frac{x^2 - 2x + 7}{x^2 + 4x - 5}$

12. A clerk's present salary is \$3500 per year. If he is to receive an increase of \$250 per year, what will be his salary t years from now?

13. Simplify.

(a) $\frac{x^2 + 2x - 8}{x^2 + 2x - 15} \times \frac{x^2 - 6x + 9}{x^2 - 4x + 4}$ (b) $\frac{ab + ac}{ab + bc} \div \frac{a^2 - ac}{b^2 - bc}$

14. Complete: $\forall_n \sum_{p=1}^{n} p(2n - p) =$

15. Prove: $\forall_x \left[-3 \neq x \neq 1 \Longrightarrow -\frac{1}{3} < \frac{2x + 2}{x^2 + 3} < 1\right]$

16. Suppose that K is an arithmetic progression. If $K_{100} = x$ and $K_{200} = y$, what is K_{300}?

17. Solve these equations.

(a) $3\sqrt{x + 2} - 3 = 4$ (b) $0.02x + 0.6 = x$

18. Simplify.

(a) $\frac{p^2 + pq}{pq - q^2} \times \frac{q^3 - pq^2}{p^3 + p^2q}$ (b) $\frac{a^2 + 2a + 1}{6a^2 - 24} \times \frac{3a^2 + 3a - 6}{a^3 + 3a^2 + 2a} \times \frac{a^2 + 4a + 4}{a^2 - 1}$

19. Suppose that 10 is the first term of an arithmetic progression and that -2 is the common difference. How many terms, starting with the first, must you take for the sum to be 0?

20. Given eight dimes and a beam balance. If at most one coin is underweight, how can you tell with two weighings whether there is an underweight coin and, if so, which?

21. Expand.

(a) $(7x + 5x^3 + 3x^2 + 2)(x^3 + 6x + 5)$ (b) $(6x^2 + 3x + 5)(5x^3 + 7x + 1)$

22. A point moves along the number line, starting at 0, in such a way that, for each n, at the beginning of the nth second it is at P_n, where, for $n > 1$,

$$P_n = P_{n-1} + (-1)^n n.$$

How far is the point from its original position after n seconds? What is the shortest total distance it can have moved during the first n seconds?

23. If a person can row 15 miles in 5 hours against a 2-mile-per-hour current, how far can he row in the same time with the current?

24. Simplify.

(a) $\dfrac{2b + y}{2cr + 3r} + \dfrac{2d + a}{2ct + 3t}$ (b) $\dfrac{1}{5x^2 + 26xy + 5y^2} + \dfrac{1}{25x^2 + 30xy + 5y^2}$

25. Suppose that f is a sequence such that, for each n, $f_n = \sum_{p=1}^{n} (22 - 3p)$.

What is the least upper bound of the range of f?

26. Prove: $\forall_n \displaystyle\sum_{p=n+1}^{2n} \frac{1}{p} = \sum_{p=1}^{2n} \frac{(-1)^{p-1}}{p}$

27. Write a quadratic equation whose roots are such that their arithmetic mean is 30 and their geometric mean is 24.

28. A point moves along the number line, starting at 1, in such a way that, for each n, at the end of the nth second it is at P_n, where

$$P_n = P_{n-1} + \frac{(-1)^n}{2^n}.$$

How far is the point from its original position after n seconds? What is the shortest total distance it can have moved during the first n seconds?

29. A 6-gallon-capacity automobile radiator is filled with an anti-freeze mixture containing 15% alcohol. How much must be drawn off and replaced by pure alcohol to have the radiator contain a mixture which is 25% alcohol?

30. Complete each of the following.

(a) For each x such that $-4 \neq x \neq 4$, $\frac{x+6}{x+4} + \frac{x-2}{x-4} = \qquad\qquad + \frac{4x}{x^2-16}$

(b) For each x such that $-6 \neq x \neq 5$, $\frac{5x+31}{x+6} - \frac{2x-9}{x-5} = \qquad\qquad - \frac{11}{x^2+x-30}$

31. Solve for 'n'.

(a) $a + \frac{b}{q-pn} = c$ (b) $a - \frac{1}{p(b-n)} = c$ (c) $\frac{p}{q}(a^2 - b^2n) = r^2$

32. The volume of a sphere varies as the cube of its radius. If three lead spheres of radii 3, 4, and 5, respectively, are melted down to make a single sphere, what is its radius?

B. Simplify.

1. $2^3 2$ [Ans: 2^4]
2. $2^3 2^2$
3. $3^0 3^5$
4. $5^1 5^0 \div 5^5$
5. $x^3 x^2$
6. $x^2 x^0$
7. $x^3 x^1 x^2$
8. $x^2 x^0 \div x^2$
9. $4^2 \cdot 5 \cdot 4^3$
10. $4^2(5 \cdot 4)^3$
11. $x^2 y x^3$
12. $x^2(yx)^3$
13. $(-6)^5(-6)^8$
14. $(-3)^4 3^9$
15. $d^4 \div d^9$
16. $19^5 19^{10}$
17. $y^3 y^0 y^9$
18. $3^7 2^3 3^2 2^6$
19. $x^7 z^3 \div (x^2 z^5)$
20. $3^{17} 3^{17}$
21. $3^{17} 3^{17} 3^{17}$
22. $5^2 9^5 \div 5^9 9$
23. $(-r)^2(-9)^3(-r)^9 9$
24. $3^{106} 3^{207}$
25. $x^{674} y^{65} x^{186}$
26. $5^2 100^3 6^3 100^2$
27. $a^2 100^4 b^6 100^3$
28. $0^5 0^0 0^6 0^8 0^{179}$
29. $(5a)^2(-5a)^3$
30. $(3x)^2(3xy)^3(xy)^2$
31. $8^5 6 3^9 0^{13} 8^{14}$
32. $x^2 y^2 x^4 \div (y^3 z^2 x^7)$
33. $ab^2c^3a^3b^2c$
34. $2.9t(2.9t)^2$
35. $(4.6p)(4.6p)(4.6p)^3$
36. $(2a)(4c)(3a)^2(4a)^3(2c^2)^5$
37. $6.7mn(6.7mn)^2(6.7mn)(6.7mn)^3$

C. Partition into sets of equivalent expressions.

5^3 fourth power of 5 $(4^2)^3$ $4^3 \times 4$ 4^4

5×5^3 4^6 $5^1 \times 5^3$ $50^3 \div 10^3$ $(2^2)^3$ $4 \times 4^2 \times 4$

5 cubed $(3^2)^3$ fifth power of 3 $4 \times 4 \times 4 \times 4 \times 4$

$5^8 \div 5^5$ $5^3 \times 5^2$ 5×5^2 the cube of 5 8^2

$4^2 \times 4^2$ $(5^2)^2$ $(5^2 \times 5^3) \div 5$ 4^3 5^4 $4^5 \div 4$

$5^8 \div 5^4$ third power of 5 5^2 3^5 $(3^3)^2$

$(5 \times 5) \times 5$ fifth power of 4 $(2^4)^2$ 4 to the fourth power

$4^8 \div 4^5$ 125 $2 \times 2 \times 2^4$ 2^8 $8^3 \div 4^3$

fourth power of 4 $3^2[(3 \times 3) \times 3]$ 3 to the fifth power

third power of the second power of the square root of 5

D. Partition into sets of equivalent expressions. [Assume that the domain of 'x' and 'y' is the set of nonzero real numbers.]

$(xxx)^3$ $(x^2)^4$ $(y^4x)^2x$ x^3y^6 x^9 x^0x^6 x^5x

$(x^4)^2$ x^6 x^3y^5y $x^6 \cdot x^3$ $x^9 \div x^3$ $x^9 \div x^0$ $x^4y^7 \div (xy)$

$(x^2)^3$ x^2y^5 $x^3(y^2)^4$ $x \cdot x^8$ $x(xx)^3x$ x^4x^2

$(x^3)^2$ $(x^5)^2$ $x^1x^3x^0x^2x^3$ $(x^1y^2)^3$ $x^5x^1 \div x^0$ $(x^3)^0(x^0)^3x^6$

$xxxyyyy^5$ $x^3y^3y^2$ $(x^2x^2)^2$ $x(x^2)^4$ $x^{12} \div x^6$ $x^9y^{12} \div (x^3y^2)^2$

x^3x^8 $(x^2)^3(x^1)^3$ $(x^3y^2)^4$ $(y^2)^3x^3y^2$ $(xy^2)^3y^0y^2$

$x^5(x^2)^4$ $x^2 \cdot x^3$ $(x^6y^4)^2$ $(xy)^0(xy)^2(xy)^3y^3 \div x^2$

$(x^3)^4y^8$ $(xy^2)^3$ $(xy)^3y^3$ x^2x^7 $(xy)^3y^5$

$x^2x^3x^4$ $x(xy)^2(y^3)^2$ x^3x^3 $x^2y^4xy^4$ $x^{12} \div (x^0x^1x^2x^3)$

INTEGRAL EXPONENTS

In the preceding exercises you reviewed some techniques you learned in Unit 4 for simplifying exponential expressions. Here are instances of three theorems on which these techniques are based:

$$7^3 \cdot 7^2 = 7^{3+2}, \qquad (5^2)^3 = 5^{2 \cdot 3}, \qquad (3 \cdot 4)^2 = 3^2 \cdot 4^2$$

Up to now we have dealt only with nonnegative integral exponents. Now we shall see how to interpret exponential expressions for the case of negative integral exponents in such a way that theorems like the three instanced above will apply. For example, we wish to define '$x^{\bar{}3}$' in such a way that

$$2^{^+3} \cdot 2^{\bar{}3} = 2^{^+3 + \bar{}3}.$$

Since $^+3 + \bar{}3 = 0$ and $2^0 = 1$, what we wish is that

$$2^{^+3} \cdot 2^{\bar{}3} = 1.$$

Since $2 \neq 0$, it follows from Exercise 6 of Part G on page 8-103 that $2^{^+3} \neq 0$. So, what we wish will be the case if and only if

$$2^{\bar{}3} = \frac{1}{2^{^+3}}.$$

And, since $^+3 = -\bar{}3$, this amounts to:

$$2^{\bar{}3} = \frac{1}{2^{-\bar{}3}}$$

More generally, for each $k < 0$, we wish that

$$2^{-k} \cdot 2^k = 2^{-k+k} = 2^0 = 1.$$

Since $2 \neq 0$ and since, for $k < 0$, $-k \in I$ and $-k > 0$, it follows, again from Exercise 6, that, for $k < 0$, $2^{-k} \neq 0$. So, our requirement is equivalent to:

$$\forall_{k < 0}\ 2^k = \frac{1}{2^{-k}}$$

The same argument applies with any nonzero number instead of 2. However, the fact that $0^{^+3} = 0$ shows that any attempt to define '$0^{\bar{}3}$' so that

$$0^{^+3} \cdot 0^{\bar{}3} = 0^{^+3 + \bar{}3} = 0^0 = 1$$

is bound to fail ['$0x = 1$' has no solution].

The preceding considerations suggest that we supplement the recursive definition of exponential sequences:

$$\begin{cases} \forall_x\, x^0 = 1 \\ \forall_x \forall_{k \geq 0}\, x^{k+1} = x^k \cdot x \end{cases}$$

by:

$$\boxed{\forall_{x \neq 0} \forall_{k < 0}\, x^k = \frac{1}{x^{-k}}}$$

Replacing 'k' by '−k', and recalling that $-k < 0$ if and only if $k > 0$ and that $--k = k$, we obtain the equivalent statement:

$$(1) \qquad \forall_{x \neq 0} \forall_{k > 0}\, x^{-k} = \frac{1}{x^k}$$

On the other hand, recalling that, for $a \neq 0$, $\frac{1}{1/a} = a$, we see that the boxed statement is also equivalent to:

$$(2) \qquad \forall_{x \neq 0} \forall_{k < 0}\, x^{-k} = \frac{1}{x^k}$$

Finally,

$$(3) \qquad a^{-0} = \frac{1}{a^0}. \qquad \text{[Explain.]}$$

Combining these three results we obtain:

<u>Theorem 154.</u>

$$\forall_{x \neq 0} \forall_k\, x^{-k} = \frac{1}{x^k}$$

<u>Examples.</u> Transform to a simple expression with nonnegative exponents.

1. $3^{-2} 3^5 3^{-9} = \frac{1}{3^2} \cdot 3^5 \cdot \frac{1}{3^9} = \frac{3^5}{3^{11}} = \frac{1}{3^6}$

2. $(x^{-2}y^3)^{-1} = \left(\frac{y^3}{x^2}\right)^{-1} = \frac{x^2}{y^3}$, $[x \neq 0 \neq y]$

3. $\frac{x^{-2}y^{-3}}{x^2} = \left(\frac{1}{x^2} \cdot \frac{1}{y^3}\right) \div x^2 = \frac{1}{x^4y^3}$, $[x \neq 0 \neq y]$

EXERCISES

A. Transform to a simple expression with nonnegative exponents.

1. $9^{-2}9^{3}9^{-5}$
2. $\dfrac{5^{-3}5^{-7}}{5^{12}}$
3. $7^{-2}7^{-3}$
4. $3^{5}3^{-5}$
5. $2^{2}2^{-8}2^{7}$
6. $3^{0}3^{-3}3^{1}$
7. $\dfrac{6^{3}\cdot 6^{-7}}{6^{2}}$
8. $\dfrac{9^{-2}9^{-5}}{9^{4}}$
9. $\dfrac{8^{3}8^{-7}}{8^{-5}}$
10. $x^{3}x^{-4}$
11. $x^{-3}x^{4}$
12. $yx^{2}y^{-2}$

B. Graph.

1. $y = 1^{k}, \ -10 \le k \le 10$
2. $y = 2^{k}, \ -4 \le k \le 4$
3. $y = \left(\frac{1}{2}\right)^{k}, \ -4 \le k \le 4$

*

4. On the same chart, sketch the graphs of the following equations for $0 \ne |x| \le 3$.

$$y = x^{-1}, \quad y = x^{-2}, \quad y = x^{-3}$$

C. Prove each of the following theorems [Theorems 152a, b, c].

Sample. $\forall_{k}\ 1^{k} = 1$ [Theorem 150a]

Solution. By Theorem 86a, $k \ge 0$ or $k < 0$. Suppose that $k \ge 0$. Then, by Exercise 1 of Part G on page 8-103, $1^{k} = 1$. Suppose that $k < 0$. Then, by definition, since $1 \ne 0$, $1^{k} = \frac{1}{1^{-k}}$. Since $-k \in I^{+}$, it follows, again from Exercise 1, that $1^{-k} = 1$. So, $1^{k} = 1/1 = 1$. Since, in both cases, $1^{k} = 1$, it follows that $\forall_{k}\ 1^{k} = 1$.

1. $\forall_{x > 0} \forall_{k}\ x^{k} > 0$
2. $\forall_{x \ne 0} \forall_{k}\ x^{k} \ne 0$
☆3. $\forall_{x > 1} \forall_{k}\ x^{k+1} > x^{k}$

LAWS OF EXPONENTS

If you worked on Exercise 3 of Part C you may have felt a need for:

$$(*) \qquad \forall_{x \neq 0} \forall_k \; x^{k+1} = x^k x$$

[Use this theorem now, to solve Exercise 3 of Part C.]

Let's prove (*).

As usual, either $k \geq 0$ or $k < 0$. In the case $k \geq 0$, it follows from the recursive definition of exponential sequences that,

$$a^{k+1} = a^k \cdot a.$$

Now, suppose that $k < 0$. It follows, by definition, for $a \neq 0$, that

$$(1) \qquad a^k \cdot a = \frac{a}{a^{-k}}.$$

Now, $-k = (-k - 1) + 1$ and, for $k < 0$, $-k - 1$ is a nonnegative integer. Hence, by the recursive definition, it follows that, for $k < 0$,

$$(2) \qquad a^{-k} = a^{(-k-1)+1} = a^{-k-1} a.$$

So, for $a \neq 0$, it follows from (1) and (2) that

$$a^k \cdot a = \frac{a}{a^{-k-1} a} = \frac{1}{a^{-k-1}}.$$

Hence, by Theorem 154, for $k < 0$ and $a \neq 0$,

$$a^k \cdot a = \frac{1}{a^{-(k+1)}} = a^{k+1}.$$

Since, for $a \neq 0$, in both cases, $a^{k+1} = a^k \cdot a$, it follows that

$$\forall_{x \neq 0} \forall_k \; x^{k+1} = x^k x.$$

We shall now make further use of (*) and one of its corollaries:

$$(**) \qquad \forall_{x \neq 0} \forall_k \; x^{k-1} = x^k / x$$

in order to prove <u>the addition law for exponents</u> [Theorem 155].

Theorem 155.

$$\forall_{x \neq 0}\ \forall_j\ \forall_k\ x^j x^k = x^{j+k}$$

We prove it inductively using Theorem 117. [Read Theorem 117 now.]

We need to prove:

(i) $a^j a^0 = a^{j+0}$

(ii) $\forall_k [a^j a^k = a^{j+k} \Rightarrow (a^j a^{k+1} = a^{j+(k+1)} \text{ and } a^j a^{k-1} = a^{j+(k-1)})]$

To establish (ii), we need to derive from the antecedent [inductive hypothesis] each component of the consequent.

Part (i):

(1)	$a^0 = 1$		[recursive definition]
(2)	$a^j a^0 = a^j \cdot 1$	[a ≠ 0]	[(1), algebra]
(3)	$= a^j$		
(4)	$= a^{j+0}$		

Part (ii):

(5)	$a^j a^i = a^{j+i}$	[a ≠ 0]	[inductive hypothesis]*
(6)	$a^{i+1} = a^i \cdot a$	[a ≠ 0]	[(*)]
(7)	$a^j \cdot a^{i+1} = a^j(a^i \cdot a)$	[a ≠ 0]	[(6), algebra]
(8)	$= (a^j a^i)a$		
(9)	$= a^{j+i} a$		[(5), (8)]
(10)	$a^{(j+i)+1} = a^{j+i} a$	[a ≠ 0]	[(*)]
(11)	$a^j \cdot a^{i+1} = a^{(j+i)+1}$		[(9), (10)]
(12)	$= a^{j+(i+1)}$		[(11), algebra]

[We have now derived the first component of the consequent of (ii). Now we get the second.]

(13) $a^{i-1} = \frac{a^i}{a}$ $\quad [a \neq 0] \quad$ [(**)]

(14) $a^j a^{i-1} = a^j \cdot \left(\frac{a^i}{a}\right)$

(15) $= \frac{a^j a^i}{a}$ $\quad$ (14)–(15): $[a \neq 0] \quad$ [(13), algebra]

(16) $= \frac{a^{j+i}}{a}$ $\quad [a \neq 0] \quad$ [(5), (15)]

(17) $a^{(j+i)-1} = \frac{a^{j+i}}{a}$ $\quad [a \neq 0] \quad$ [(**)]

(18) $a^j a^{i-1} = a^{(j+i)-1}$ $\quad$ [(16), (17)]

(19) $= a^{j+(i-1)}$ $\quad$ [(18), algebra]

(20) $\forall_k [a^j a^k = a^{j+k} \Rightarrow (a^j a^{k+1} = a^{j+(k+1)}$ and $a^j a^{k-1} = a^{j+(k-1)})]$ $\quad$ [(12),(19); *(5)]

Part (iii):

(21) $\forall_k\ a^j a^k = a^{j+k}$ $\quad$ [(4),(20), PMI]

---

(22) $\forall_{x \neq 0}\ \forall_j\ \forall_k\ x^j x^k = x^{j+k}$ $\quad$ [(1) - (21)]

The reference 'PMI' for (21) refers to Theorem 117.

As a corollary to the theorem just proved we have the subtraction law for exponents:

> Theorem 156.
>
> $$\forall_{x \neq 0}\ \forall_j\ \forall_k\ \frac{x^j}{x^k} = x^{j-k}$$

In a similar manner, using the two theorems just proved instead of (*) and (**), one can prove the second of the three fundamental theorems on exponents, the multiplication law for exponents:

> Theorem 157.
>
> $$\forall_{x \neq 0}\ \forall_j\ \forall_k\ (x^j)^k = x^{jk}$$

The third of the three fundamental theorems on exponents--the distributive law for exponentiation over multiplication:

Theorem 158.

$$\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \; (xy)^k = x^k y^k$$

--can also be proved inductively using Theorem 117. An alternative method of proof is to derive the case $k \geq 0$ from Theorem 145 as you did in Exercise 1 of Part B on page 8-101 and to take care of the case $k < 0$ as you did in the exercises of Part C on page 8-115.

As a corollary of Theorem 158 we have:

Theorem 159.

$$\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}$$

[Suggest an appropriate name for this law.] Also:

Theorem 160.

$$\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \left(\frac{x}{y}\right)^{-k} = \left(\frac{y}{x}\right)^k$$

EXERCISES

A. Prove.

1. Theorem 156 ☆2. Theorem 157 ☆3. Theorem 158

4. Theorem 159 5. Theorem 160

* * *

You have had considerable practice in manipulating expressions containing exponent symbols for nonnegative integers. Now you have the theorems which enable you to extend this skill to expressions containing exponent symbols for any integers. Usually, a simplification of such an expression consists in transforming it into a simpler one which does not contain exponent symbols with oppositing signs. In

doing such exercises, it is tacitly assumed that the variables involved do not have values which lead to division by 0.

Examples.

1. $x^{-3}x^{5}x^{-4} = x^{-2} = \frac{1}{x^{2}}$

2. $\frac{a^{-3}b^{5}c^{-4}}{a^{-2}b^{-7}c^{5}} = a^{-1}b^{12}c^{-9} = \frac{b^{12}}{ac^{9}}$

3. $\frac{a^{-m}+b^{-n}}{a^{-m}-b^{-n}} = \frac{(a^{-m}+b^{-n})(a^{m}b^{n})}{(a^{-m}-b^{-n})(a^{m}b^{n})} = \frac{b^{n}+a^{m}}{b^{n}-a^{m}}$

4. $(x^{-1}+y^{-1})(x^{-1}-y^{-1}) = x^{-2}-y^{-2} = \frac{y^{2}-x^{2}}{x^{2}y^{2}}$

5. $(3a^{2}b^{-1})^{-2}(3^{2}a^{-3}b^{-4})^{-5} = 3^{-2}a^{-4}b^{2}3^{-10}a^{15}b^{20} = \frac{a^{11}b^{22}}{3^{12}}$

6. $\frac{30^{-3}25^{-4}}{10^{-6}15^{-3}} = \frac{(2^{-3}3^{-3}5^{-3})5^{-8}}{(2^{-6}5^{-6})(3^{-3}5^{-3})} = \frac{2^{-3}\cdot\cancel{3^{-3}}\cdot\cancel{5^{-3}}\cdot5^{-8}}{2^{-6}\cdot\cancel{3^{-3}}\cdot\cancel{5^{-3}}\cdot5^{-6}} = \frac{2^{3}}{5^{2}}$

* * *

B. Simplify.

1. $\frac{1}{2^{-2}}$

2. $2^{-5}2^{5}$

3. $2^{-3}+2^{3}$

4. $(-2)^{-3}$

5. $(-2)^{-3}(-2)^{-3}$

6. 10^{-6}

7. $10^{3}10^{-6}$

8. $(4^{2}8^{-8})^{0}$

9. $5^{-30}\div 5^{-22}$

10. $\frac{(-2)^{-3}}{(-2)^{-5}}$

11. $\frac{(-3)^{2}}{(-3)^{5}}$

12. $\frac{(-5)^{2}}{2^{-1}+4^{-1}}$

13. $\frac{\left(\frac{1}{2}\right)^{-3}\left(\frac{1}{3}\right)^{-3}}{\left(\frac{1}{3}\right)^{-3}\left(\frac{1}{2}\right)^{-3}}$

14. $\frac{\left(\frac{2}{5}\right)^{-2}\left(\frac{1}{3}\right)^{4}}{\left(\frac{1}{5}\right)^{-3}\left(\frac{2}{3}\right)^{2}}$

15. $\frac{24^{-3}\times 48^{3}}{36^{-2}\times 6^{4}}$

16. $\frac{8^{-2}12^{-5}20^{-2}}{4^{-3}10^{2}15^{-2}}$

17. $\frac{(3+2)^{-2}}{(3-2)^{-2}}$

18. $\frac{2^{-3}+3^{-2}}{2^{-3}-3^{-2}}$

19. $x^{-2}x^{-3}$

20. $y^{-1}y^{-5}$

21. $x^{2}x^{-1}$

22. $r^{-3}r^{4}$

23. $a^{-m}\cdot a^{3}$

24. $a^{2}a^{k}$

25. $y^{2m}y^{-m}$

26. $k^{-2}k^{0}$

27. $y^{-5}y$

28. $z^2 \cdot z^{-7}$

29. $r^5 \div r^3$

30. $r^3 \div r^5$

31. $\dfrac{a^3}{a^{-4}}$

32. $\dfrac{n^2}{n^{-6}}$

33. $\dfrac{m^{-5}}{m^3}$

34. $\dfrac{z^3}{z^{-m}}$

35. $\dfrac{t^{3m}}{t^m}$

36. $\dfrac{x^{2m}}{x^{-3m}}$

37. $\dfrac{5^{2n+1}}{5^{2n+2}}$

38. $\dfrac{m^0}{m^{-3}}$

39. $\dfrac{b^{-2}}{a^{-5}}$

40. $\dfrac{t^{-3}}{t^{-4}}$

41. $\dfrac{2x^{-6}}{y^7}$

42. $\dfrac{c^2}{k^{-3}}$

43. $\dfrac{1}{3s^{-2}}$

44. $\dfrac{1}{5n^{-3}}$

45. $\dfrac{3x^{-3}}{4y^{-2}}$

46. $\dfrac{2u^{-2}}{3v^{-5}}$

47. $\dfrac{a^{-2}b^3}{a^3b^{-2}}$

48. $\dfrac{xy^{-4}}{x^{-5}y^3}$

49. $\dfrac{rs^{-2}t^3}{r^{-4}s^5t^{-6}}$

50. $\dfrac{a^{-1}b}{b^{-4}a^3}$

51. $\dfrac{3^{-1}a^{-1}}{6^{-1}a^3b^{-2}}$

52. $\dfrac{5x}{x^{-1}} - \dfrac{3}{x^2}$

53. $\dfrac{3y}{z^{-1}} + \dfrac{2z}{y^{-1}}$

54. $\dfrac{t^{-3}+s^{-3}}{s^{-3}t^{-3}}$

55. $\dfrac{5^2x^{-3}y^4z^{-5}}{5^{-3}x^{-1}z^2}$

56. $\dfrac{3^{-2}a^{-4}b^{-1}c^{-3}}{6^{-3}a^4b^{-2}c^0}$

57. $\dfrac{12^{-2}x^{-1}y^{-3}z^2}{4^{-3}x^{-2}y^2z^{-3}}$

58. $(2x^{-1}y^{-2})^3(3x^{-2}y^{-1})^{-2}$

59. $(3^{-2}x^{-3}y^{-1})^{-2}(3^5x^{-4}y^{-3})^{-3}$

60. $(x^{-1} + y^{-1})^2 - (x^{-1} - y^{-1})^2$

61. $(a + b)^{-2} + 3(a + b)^{-1}$

62. $\dfrac{x^3y^{-2}z^4}{x^{-3}y^2z^0}$

63. $\dfrac{x^4 + x^{-2}}{x^3 + x^{-1}}$

64. $\left(\dfrac{x}{y}\right)^{-2}\left(\dfrac{x}{y}\right)^2$

65. $(x + y + z^{-1})z$

66. $\dfrac{(zx+xy)^{-1}x^2}{(xy)^0}$

67. $x^2y^{-3}z^4 + x^2y^{-4}z^3$

68. $\dfrac{x^2(-y)^3}{(-z)^0}$

69. $\dfrac{x^{-1} + a^{-1}}{x^{-2} - a^{-2}}$

70. $(u+v)^2(u^{-1}+v^{-1})^{-2}$

71. $\dfrac{3^{m-5} - 3 \cdot 3^m}{3^{m+2}}$

72. $(x+y)^{-2}(x^{-1}+y^{-1})^2$

73. $\dfrac{x^k + x^{2k}y^{-k}}{x^ky^{-k}}$

74. $\dfrac{x^2 - y^2}{x^{-1} - y^{-1}}$

75. $\dfrac{x^{2k} - y^{-2k}}{x^k - y^{-k}}$

76. $\dfrac{x^{-k} - 1}{x^{-2k} - 2x^{-k} + 1}$

77. $\dfrac{(3xy)^m(4x^2y^{-2})^{-n}}{(5x^3y)^n(2xy^3)^{-m}}$

78. $\dfrac{(3st^2)^{-j}(5s^3t^{-3})^{2j}}{(5s^{-3}t^{-2})^j(7s^5t^{-1})^{-3j}}$

79. $\dfrac{x^m x^p x^{-q}}{x^m x^{-p} x^{p+q}}$

80. $\dfrac{a^{-j} a^{-k} a^{-j}}{a^{2j-k+i}}$

81. $\dfrac{\left[\left(t^{m-n} \cdot t^{n-p}\right)^q\right]\left(\dfrac{t^m}{t^p}\right)^p}{\left(t^q t^p\right)^m \div \left(t^q t^p\right)^p}$

82. $\left(\dfrac{x^k}{x^j}\right)^{j+k}\left(\dfrac{x^j}{x^m}\right)^{m+j}\left(\dfrac{x^m}{x^k}\right)^{k+m}$

83. $\dfrac{(bc)^{mn}(ca)^{np}(ab)^{pm}}{\left(b^{m-1}c^{n-1}\right)^p\left(c^{n-1}a^{p-1}\right)^m\left(a^{p-1}b^{m-1}\right)^n}$

C. Transform these expressions into expressions which do not contain fractions or division signs.

1. $\dfrac{1}{x^2}$ 2. $\dfrac{1}{x^3x^4}$ 3. $\dfrac{1}{ab^{-2}}$ 4. $\dfrac{5}{ay^{-3}}$

5. $\dfrac{mn}{p}$ 6. $\dfrac{x}{yz}$ 7. $\dfrac{1}{2^n}$ 8. $\left(\dfrac{1}{4}\right)^m$

9. $\dfrac{x^{-2}}{y^{-2}z^2}$ 10. $\dfrac{5}{s^3t^{-3}}$ 11. $\dfrac{3x^2y}{x^{-1}y^{-3}}$ 12. $\dfrac{-9pq^2}{p^{-3}q^{-4}}$

13. $\dfrac{12xy}{2^{-2}x^{-2}y^{-2}}$ 14. $\dfrac{a^2b^{-3}}{2^{-4}a^{-4}b^{-5}}$ 15. $\dfrac{2ab}{a+b}$ 16. $\dfrac{a+b}{2ab}$

D. Solve.

Sample. $2^{3k+5} = 0.5$

Solution. Since $0.5 = 2^{-1}$, one solution of the given equation is the root of '$3k + 5 = -1$'. So, one solution is -2. Is there another?

Your work in Exercise 2 of Part B on page 8-115 and your general knowledge of exponential sequences is probably enough to convince you that if $2^{3k+5} = 2^{-1}$ then $3k + 5 = -1$. So, -2 is the only solution of the given equation. But, to get on firm ground, let's prove:

Theorem 161.

$$\forall_{x>0}\ \forall_j\ \forall_k\ [x^j = x^k \Longleftrightarrow (x = 1 \text{ or } j = k)]$$

For $a > 0$, $a \neq 0$, and, using Theorems 152b and 156, it follows that

$$a^j = a^k \iff a^{j-k} = 1.$$

By Theorem 154, it follows that $a^{j-k} = 1$ if and only if $a^{k-j} = 1$. Since, for any j and k, either $j - k \geq 0$ or $k - j \geq 0$, Theorem 161 is equivalent to:

$$(*) \qquad \forall_{x > 0} \forall_{i \geq 0} [x^i = 1 \iff (x = 1 \text{ or } i = 0)]$$

Now, by Theorem 153, for any a and any $i \geq 0$,

$$a^i = 1 \iff (a - 1) \sum_{p=1}^{i} a^{p-1} = 0$$

--that is [by the 0-product theorem and the pm0],

$$a^i = 1 \iff a = 1 \text{ or } \sum_{p=1}^{i} a^{p-1} = 0.$$

By Theorem 152a, for $a > 0$, and for any i, $a^{i-1} > 0$. Using Theorem 143, it follows that, for $i > 0$, $\sum_{p=1}^{i} a^{p-1} > 0$. Since $\sum_{p=1}^{0} a^{p-1} = 0$, it follows that, for $i \geq 0$,

$$\sum_{p=1}^{i} a^{p-1} = 0 \iff i = 0.$$

So, for $a > 0$ and $i \geq 0$,

$$a^i = 1 \iff a = 1 \text{ or } i = 0.$$

This completes the proof of (*) and, so, of Theorem 161.

1. $2^{4k} = 16$ 2. $3^{8-k} = \frac{1}{3}$ 3. $1^k = 1$ 4. $1^k = 2$

5. $4^{k-1} = 8^{1+k}$ 6. $\left(\frac{1}{4}\right)^{k-1} = \left(\frac{1}{8}\right)^{1+k}$ 7. $30^{k+1} = 90^{k+1}$

8. $\begin{cases} 4^k = \left(\frac{1}{2}\right)^{j-1} \\ 27^k = 3^{9+j} \end{cases}$ 9. $\begin{cases} \left(2^{i-2}\right)\left(\frac{1}{2}\right)^{j-1} = 1 \\ 2^{i-2} = \left(\frac{1}{2}\right)^{j-1} \end{cases}$

10. $2^k + 2^{-k} = \frac{17}{4}$

☆11. $(2^k + 2^{-k})^2 = 4^k + 4^{-k}$

☆12. $8(4^k + 4^{-k}) - 54(2^k + 2^{-k}) + 101 = 0$

☆E. 1.

2	7	6
9	5	1
4	3	8

The figure on the left is a magic addition-square. The sum of the numbers listed in each row is the sum of the numbers listed in each column and in each diagonal. Construct a magic multiplication-square.

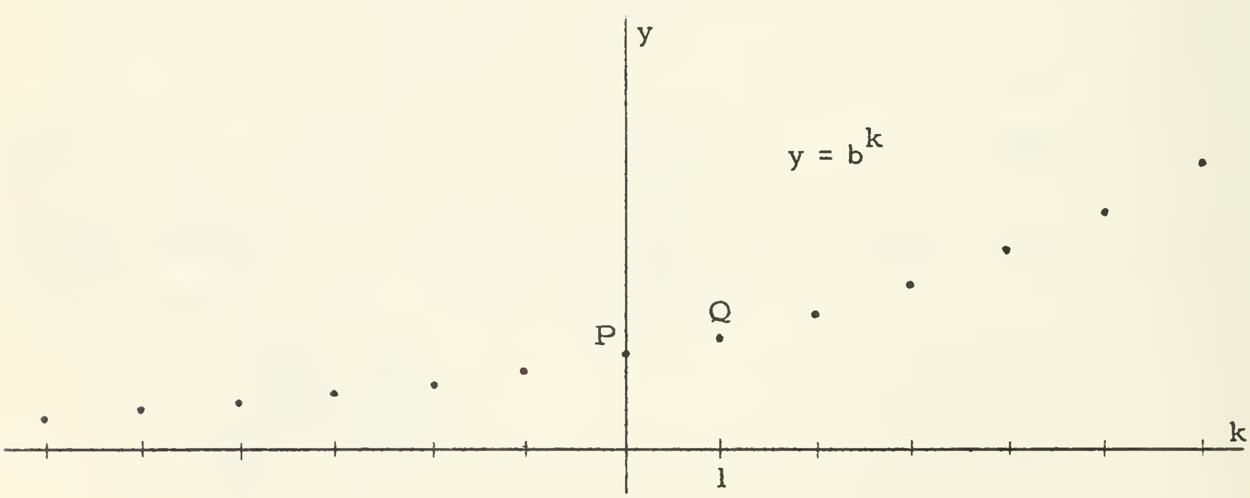
2. Put copies of the same digit in the blanks to make the sentence true.

$$2^5 \cdot \underline{\quad}^2 = 25\underline{\quad}2$$

F. 1. (a) Compute the first five terms of the exponential sequence with base 1.0001.

(b) Do you think that there is a term of the sequence which is greater than 100? Greater than 10? Greater than 2?

2. The chart below shows the graph of an exponential sequence whose base is some number b. [The function, part of whose graph is shown in the chart, is not a sequence because its domain is I. It is often called 'a two-way sequence'. Explain.]



(a) What are the coordinates of P?

(b) How can you tell that the base b is not 0? Can you tell whether b is positive or negative?

(c) How can you tell that the base b is not 1? Can you tell whether it is less than 1 or greater than 1?

(d) Sketch the graph of an exponential sequence whose base is between 0 and 1.

(e) Notice that all the points of the graph are either on or above the straight line through P and Q. What is the slope of this line and what is its y-intercept?

(f) The observation mentioned in (e) suggests the theorem:

$$\forall_{b > 1} \forall_k \; b^k \geq 1 + (b - 1)k$$

This theorem is related to another called Bernoulli's Inequality:

$$\forall_{x \geq -1} \forall_{k \geq 0} \; (1 + x)^k \geq 1 + kx \qquad \text{[Theorem 162]}$$

Prove Bernoulli's Inequality by induction.

☆(g) Prove: $\forall_{x > 0} \forall_{n > 1} \; (1 + x)^n > 1 + nx$.

3. (a) Use <u>Bernoulli's Inequality</u> to name one term [don't compute it] of the exponential sequence with base 1.0001 which is greater than 1000001.

(b) Name a term greater than 10^{10} when the base of the exponential sequence is 4/3.

(c) Name a term greater than c when the base of the exponential sequence is $b > 1$.

(d) Prove:

$$\forall_{x > 1} \forall_y \forall_n \; [n \geq \frac{y}{x - 1} \Longrightarrow x^n > y] \qquad \text{[Theorem 163]}$$

4. (a) Do you think that there is a term of the exponential sequence with base 0.9999 which is less than 1/1000000?

(b) Prove: $\forall_{x \neq 0} \; [\frac{1}{x} > 1 \Longleftrightarrow 0 < x < 1]$ [Theorem 164]

(c) Prove: $\forall_{x \neq 1} \forall_{y > 0} \forall_n \left[\left(0 < x < 1 \text{ and } n \geq \frac{1}{y(1-x)}\right) \Rightarrow x^n < y\right]$ [Theorem 165]

[Hint. Use Theorems 163 and 164. (It is also possible to prove Theorem 165 by induction.)]

(d) Name one term in the exponential sequence with base 0.9999 which is less than 1/1000000. Less than 1/4375961.

(e) Pick a very small positive number. Can you find a term in the exponential sequence with base 0.9999 which is smaller? Can you find a term such that all terms after it are smaller than the number you picked?

(f) Repeat (e) for a nonpositive number.

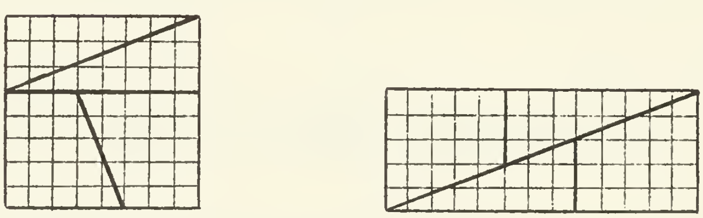
☆G. The Fibonacci sequence f may be defined by:

$$\begin{cases} f_{-1} = 1 \\ f_0 = 0 \\ \forall_n \ f_n = f_{n-2} + f_{n-1} \end{cases}$$

[See Part E on page 8-24 and Part D on page 8-46.]

1. Prove: $\forall_{k \geq 0} \ f_{k+1} f_{k-1} - f_k^{\,2} = (-1)^k$

2. You may be acquainted with the following puzzle:



Cut an 8×8 square as shown, and reassemble the pieces as shown in the right-hand picture.

When one does this [do it], it appears that a square region

with area-measure 64 can be cut and pieced together to form a rectangular region with area-measure 65. The problem is, of course, to account for the extra square unit of area. Do so.

3. Notice that the side-measure of the square in Exercise 2 is f_6. From this hint find integers m, n, and p [other than 8, 13, and 5] such that an m × m square region can be cut up and pieced together so as to appear to cover an n × p rectangular region whose area-measure differs from that of the square by 1.

* * *

☆Let's try to find an explicit definition for the Fibonacci sequence. One standard technique for solving problems like this is, first, to find lots of sequences which satisfy the <u>recursion equation</u>--in this case, the equation:

(*) $$a_n = a_{n-2} + a_{n-1}$$

--and, then, to look among these sequences for one which satisfies the <u>initial conditions</u>--in this case:

(**) $$a_{-1} = 1 \text{ and: } a_0 = 0$$

There is a standard method for finding sequences which satisfy recursion equations like (*). To begin with, we look for a real number x such that if, for each $k \geq -1$, $a_k = x^k$ then a satisfies the recursion equation. Applied to (*), this leads to the equation:

$$x^n = x^{n-2} + x^{n-1},$$

which is equivalent to:

$$x^{n-2}(x^2 - x - 1) = 0$$

--that is, to:

$$x^{n-2} = 0 \text{ or } x^2 - x - 1 = 0$$

So, if r is either root of the equation '$x^2 - x - 1 = 0$' then the sequence a such that, for each $k \geq -1$, $a_k = r^k$ is a solution of (*). This gives us two solutions of (*) [Find them.]. Unfortunately, neither of these solutions satisfies the initial conditions (**) [Check this.].

Fortunately, there is a way out of this. Suppose that a′ and a″ are two solutions of (*)--that is, suppose that, for each n,

$$a'_n = a'_{n-2} + a'_{n-1}$$

and

$$a''_n = a''_{n-2} + a''_{n-1}.$$

It is easy to see that if x and y are constant sequences then xa′ + ya″ also satisfies (*)--that is, for each n,

$$(xa' + ya'')_n = (xa' + ya'')_{n-2} + (xa' + ya'')_{n-1}. \qquad \text{[Explain.]}$$

Since we have already found two sequences which satisfy (*)--a′ and a″, where

$$a'_k = \left(\frac{1+\sqrt{5}}{2}\right)^k \quad \text{and} \quad a''_k = \left(\frac{1-\sqrt{5}}{2}\right)^k$$

--we can now find lots of others. In fact, for any real numbers x and y, the sequence a such that, for $k \geq -1$,

$$a_k = x\left(\frac{1+\sqrt{5}}{2}\right)^k + y\left(\frac{1-\sqrt{5}}{2}\right)^k$$

is a solution of (*).

* * *

☆H. 1. Prove: $\forall_{k \geq -1}\ f_k = \dfrac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}}$

2. Use the result of Exercise 1 to compute f_4.

3. Use the procedure illustrated above to find an explicit definition for the sequence a defined recursively by:

$$\begin{cases} a_{-1} = 0 \\ a_0 = 1 \\ \forall_n\ a_n = \dfrac{a_{n-2} + a_{n-1}}{2} \end{cases}$$

4. Generalize the result of Exercise 3 by finding an explicit definition for the sequence whose nth term is obtained by averaging the two preceding terms, starting with any two numbers a_{-1} and a_0.

GEOMETRIC PROGRESSIONS

For each real number r, the sequence e defined by:

$$\forall_n \; e_n = r^{n-1}$$

is the exponential sequence with base r. It can be defined recursively by:

$$\begin{cases} e_1 = 1 \\ \forall_n \; e_{n+1} = e_n r \end{cases}$$

A sequence a such that, for some r,

$$(*) \quad \begin{cases} a_1 \neq 0 \\ \forall_n \; a_{n+1} = a_n r \end{cases}$$

is called <u>a geometric progression</u>. Evidently, each exponential sequence is a geometric progression. We shall see, shortly, that each geometric progression is "almost" an exponential sequence [Exercise 1 of Part C on page 8-133].

Listed below are the first few terms of some sequences. Decide which of the sequences could be geometric progressions. For each of these, find an appropriate number r.

(1) 2, 4, 8, 16, 32, 64, ...

(2) 2, 4, 6, 8, 10, 12, ...

(3) $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

(4) −1, 3, −9, 27, −81, 243, ...

(5) 6, 0, 0, 0, 0, ...

(6) 0, 1, 2, 4, 8, 16, ...

(7) $1, 1\frac{1}{2}, 2\frac{1}{4}, 3\frac{3}{8}, 5\frac{1}{16}, 7\frac{19}{32}, \ldots$

(8) 0, 0, 0, 0, 0, ...

(9) 2, 2, 2, 2, 2, ...

Were any of these sequences arithmetic progressions? If so, tell which and give the common difference.

It follows from (*) that if a is a GP then the number r such that, for each n, $a_{n+1} = a_n \cdot r$ is a_2/a_1. This number is called <u>the common ratio</u> of the GP. If the common ratio is not 0 then [as you can easily prove] no term of the GP is 0. Moreover, for $r \neq 0$,

$$\forall_n \frac{a_{n+1}}{a_n} = r. \qquad \text{[Theorem 167b]}$$

[What can you say about the terms of a GP whose common ratio is 0?]

EXERCISES

<u>A</u>. 1. Fill in the blanks in the following lists of terms of GPs. [In some cases there is more than one solution.]

(a) 1, 2, , 8, , , ...

(b) 3, , , 81, , , ...

(c) -2, , $-\frac{1}{2}$, , , ...

(d) -9, , , $\frac{1}{3}$, , , ...

(e) 3, $3\sqrt{2}$, , , , , ...

(f) $\sqrt{3}$, , , 9, , , ...

(g) $\sqrt{3}$, , , -9, , , ...

(h) -3, , , , -3, , ...

(i) π, , , , 0, , ...

(j) , , , , , 0, ...

2. In filling the blanks between '3' and '81' in part (b) of Exercise 1 you inserted two <u>geometric means</u> between 3 and 81. In part (c) you inserted one geometric mean between -2 and $-\frac{1}{2}$ [and found that there were two ways in which you could do this].

(a) Insert two geometric means between 2 and 250.

(b) Insert three geometric means between 1 and 256.

(c) Insert four geometric means between -1 and 32.

(d) Insert three geometric means between -1 and 8.

3. When can you insert three geometric means between two numbers? Any odd number of geometric means?

4. If three positive numbers are consecutive terms of a GP then the second is called <u>the geometric mean</u> of the first and third.

   (a) Find the geometric mean of 2 and 4.

   (b) Find the geometric mean of 1 and 9.

   (c) Prove the following generalization:

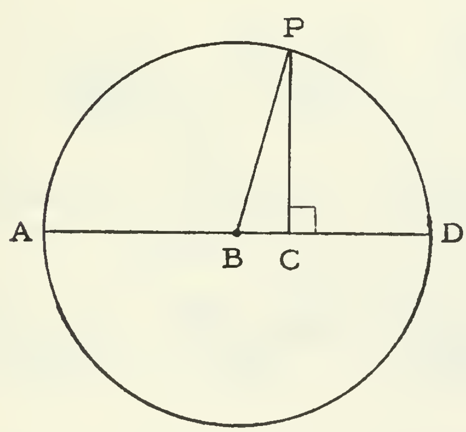
$$\forall_{x>0} \forall_{y>0} \text{ the geometric mean of } x \text{ and } y \text{ is } \sqrt{xy}$$

5. Suppose that the passing grade in your class is 65 and that you have taken two examinations on one of which your grade was 40, on the other 90. When ''averaging'' these grades, would you rather that your teacher used their arithmetic mean [page 8-74] or their geometric mean?

6. What theorem which you have previously studied on inequations justifies your answer to Exercise 5?

7. Prove that the arithmetic mean of any two positive numbers is always greater than their geometric mean:

$$\forall_{x>0} \forall_{y>0} \text{ if } x \neq y \text{ then } \frac{x+y}{2} > \sqrt{xy} \qquad \text{[Theorem 166]}$$

8. 

Show geometrically that the arithmetic mean of two positive numbers [AC and CD] is greater than their geometric mean.

9. Complete:

$$\forall_{x>0} \forall_{y>0} \text{ the geometric mean of } \frac{x+y}{2} \text{ and } \frac{2xy}{x+y} \text{ is}$$

☆B. Recall the dividing-and-averaging method, which you studied in Unit 3, for finding approximations to principal square roots of positive numbers. The idea was that if y_1 is a fairly good approximation to $\sqrt{a}$, and $y_2 = \dfrac{y_1 + \frac{a}{y_1}}{2}$, then y_2 is a better approximation to $\sqrt{a}$. Note that y_2 is the arithmetic mean of y_1 and $\frac{a}{y_1}$ and that $\sqrt{a}$ is their geometric mean. It follows from Theorem 166 that,

$$\text{if } y_1 \neq \sqrt{a} \text{ then } y_2 > \sqrt{a}.$$

1. Prove the algebra theorem:

$$\forall_{x>0} \forall_{y>0} \; \frac{x+y}{2} - \sqrt{xy} = \frac{(\sqrt{x} - \sqrt{y})^2}{2}$$

and use it to show that [for $y_1 > 0$]

$$(*) \qquad y_2 - \sqrt{a} = \frac{(y_1 - \sqrt{a})^2}{2y_1}.$$

2. The result of Exercise 1 can be used to estimate the error in the approximation y_2 when one knows an estimate for the error in the approximation y_1. For example, if $y_1 \geq 5$ and $|y_1 - \sqrt{a}| < 10^{-n}$ then it follows that $0 < y_2 - \sqrt{a} < 10^{-(2n+1)}$. Proving the following theorems will show another way to estimate the error in y_2.

(a) $\forall_{x>0} \forall_{y>0} \; [x > y \Rightarrow 4y < (\sqrt{x} + \sqrt{y})^2 < 4x]$

(b) $\forall_{x>0} \forall_{y>0} \; [x > y \Rightarrow \dfrac{(x-y)^2}{8x} < \dfrac{(\sqrt{x} - \sqrt{y})^2}{2} < \dfrac{(x-y)^2}{8y}]$

(c) $\forall_{y_1>0} \forall_{a>0} \; [y_1^2 > a \Rightarrow \dfrac{(y_1^2 - a)^2}{8y_1^3} < \dfrac{(y_1 - \sqrt{a})^2}{2y_1} < \dfrac{(y_1^2 - a)^2}{8ay_1}]$

(d) $\forall_{y_1>0} \forall_{a>0} \; [y_1^2 < a \Rightarrow \dfrac{(y_1^2 - a)^2}{8ay_1} < \dfrac{(y_1 - \sqrt{a})^2}{2y_1} < \dfrac{(y_1^2 - a)^2}{8y_1^3}]$

3. Combining (*) of Exercise 1 with (c) and (d) of Exercise 2 shows

that, in the dividing-and-averaging procedure, the error in y_2 is always between $\frac{(y_1{}^2 - a)^2}{8ay_1}$ and $\frac{(y_1{}^2 - a)^2}{8y_1{}^3}$. Show that, for each x near 0, $1 + \frac{x}{2}$ is a good approximation for $\sqrt{1 + x}$ by showing that $(1 + \frac{x}{2}) - \sqrt{1 + x}$ is between $\frac{x^2}{8}$ and $\frac{x^2}{8(1 + x)}$.

C. Suppose that a is a geometric progression with common ratio r, and, for each n, s_n is the sum of its first n terms. Prove each of the following theorems.

1. $\forall_n\ a_n = a_1 r^{n-1}$ [Theorem 167a]

2. $\forall_{r \neq 1}\ \forall_n\ s_n = \frac{a_1(1 - r^n)}{1 - r}$ [Theorem 167c]

   [Hint. You can use a theorem you have already proved on continued sums of terms of exponential sequences.]

3. $\forall_{r \neq 1}\ \forall_n\ s_n = \frac{a_1 - a_n r}{1 - r}$ [Theorem 167d]

4. Complete: $r = 1 \Longrightarrow \forall_n\ s_n =$

*

5. Show that a sequence is both an AP and a GP if and only if it is a nonzero constant sequence.

* * *

Example 1. Find the twelfth term of the geometric progression

3, 6, 12, 24, ...

Solution. The theorem of Exercise 1 of Part C provides a formula for this problem.

$$a_n = a_1 r^{n-1}$$

$$a_{12} = 3 \cdot 2^{12-1}$$

$$= 3 \cdot 2048 = 6144$$

Example 2. Find the sum of the first twelve terms of the geometric progression

$$6,\ 2,\ \frac{2}{3},\ \frac{2}{9},\ \ldots\ .$$

Solution. From Exercise 2 of Part C we get the formula:

$$s_n = \frac{a_1(1 - r^n)}{1 - r}$$

So, in this example,

$$s_{12} = \frac{6\left[1 - \left(\frac{1}{3}\right)^{12}\right]}{1 - \frac{1}{3}} = \frac{6\left[1 - \frac{1}{3^{12}}\right]}{\frac{2}{3}}$$

$$= 9\left(\frac{3^{12} - 1}{3^{12}}\right) = \frac{531440}{59049}.$$

Example 3. Find the sum of the first twelve terms of the geometric progression in Example 1.

Solution. Since we know the twelfth term, we use a formula obtained from Exercise 3 of Part C.

$$s_n = \frac{a_1 - a_n r}{1 - r}$$

So, in this example,

$$s_{12} = \frac{3 - 6144 \cdot 2}{1 - 2} = \frac{-12285}{-1} = 12285$$

* * *

D. 1. What is the ninth term of the geometric progression

$$\frac{1}{32},\ \frac{1}{16},\ \frac{1}{8},\ \ldots\ ?$$

2. Find the sum of the first ten terms of the geometric progression

$$1,\ -\frac{1}{2},\ \frac{1}{4},\ -\frac{1}{8},\ \ldots\ .$$

3. If the first term of a GP is 1 and the common ratio is -1, what is the twentieth term and what is the sum of the first twenty terms?

4. If the eighth term of a GP is $\frac{1}{2}$ and if the common ratio is $\frac{1}{2}$, what is the first term?

5. Find the twentieth term of the geometric progression $3\sqrt{3}$, 9, $9\sqrt{3}$, ...

6. If a is a GP such that $a_3 = 2^{-3}$ and $a_6 = 2^{-9}$, what is a_1?

7. Find the sum of the first 10 terms of the geometric progression 1000, 100, 10, ...

8. Find the sum of the first 15 terms of the GP whose second term is 1 and whose fifth term is 3^{-3}.

9. What is the sum of the first 10 terms of the geometric progression t, s^2t^2, s^4t^3, ...?

10. How many terms of the geometric progression 1, 3, 9, ... are between 10^4 and 10^5?

11. How many terms of the geometric progression 1, $\frac{1}{3}$, $\frac{1}{9}$, ... are between 10^{-4} and 10^{-5}?

12. What is the smallest number of consecutive terms of the geometric progression 1, $\frac{1}{3}$, $\frac{1}{9}$, ... whose sum exceeds 1.5?

13. Suppose a is the geometric progression such that $a_1 = \frac{27}{10^2}$ and

$$\forall_n \; a_{n+1} = \frac{a_n}{10^2}.$$

(a) Find s_3, s_4, s_5.

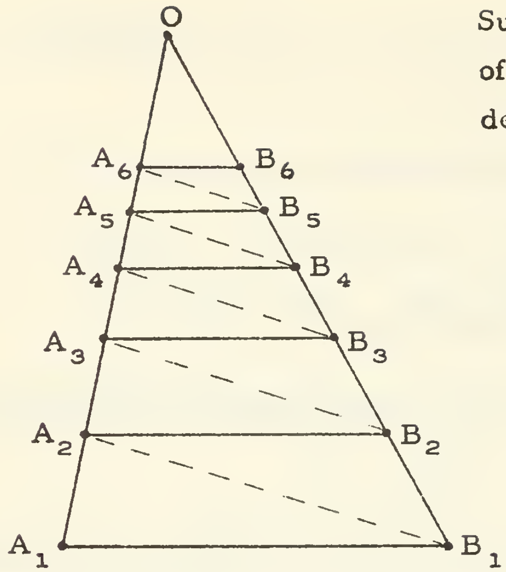
(b) Prove: $\forall_n \; s_n < \frac{3}{11}$

(c) Find $\frac{3}{11} - s_3$, $\frac{3}{11} - s_4$, $\frac{3}{11} - s_5$.

(d) Find the smallest m such that $\frac{3}{11} - s_m < 10^{-100}$.

(e) Prove: $\forall_m \forall_{n>m} \; \frac{3}{11} - s_n < \frac{3}{11} - s_m$.

E. 1.



Suppose that, for ΔA_1OB_1, a sequence of segments parallel to $\overline{A_1B_1}$ is defined by the following:

$$\begin{cases} \overline{A_2B_2} \parallel \overline{A_1B_1} \\ \forall_n \left(\overline{B_{n+1}A_{n+2}} \parallel \overline{B_nA_{n+1}} \text{ and } \overline{A_{n+2}B_{n+2}} \parallel \overline{A_{n+1}B_{n+1}}\right) \end{cases}$$

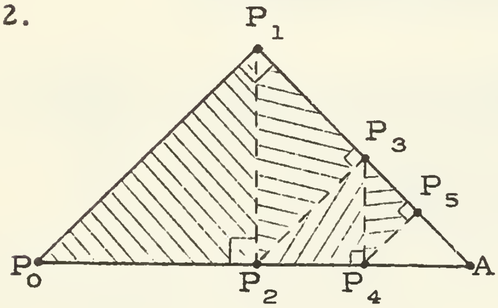
(a) Show that A_1B_1, A_2B_2, A_3B_3, ... are consecutive terms of a geometric progression.

(b) Suppose that, for each n, $\{S_n\} = \overrightarrow{A_{n+1}B_n} \cap \overrightarrow{A_1B_1}$. Show that, for each n,

$$\sum_{p=1}^{n} A_pB_p = A_1S_n .$$

(c) Suppose that S is the point of $\overrightarrow{A_1B_1}$ such that $\overrightarrow{OS} \parallel \overline{A_2B_1}$. Prove: $\forall_n\ A_1S_n < A_1S$

2.

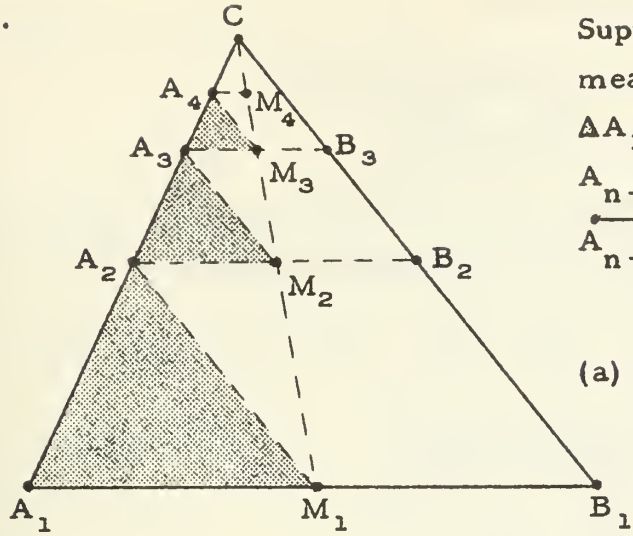


Suppose that ΔP_0P_1A is a right triangle with $P_0P_1 = P_1A$ and area-measure 1. Triangular regions are formed by dropping perpendiculars from P_1, P_2, P_3, P_4, ...

(a) $\forall_n \sum_{k=1}^{n} K(\Delta P_{k-1}P_kP_{k+1}) =$

(b) Find the smallest x such that $\forall_n \sum_{k=1}^{n} K(\Delta P_{k-1}P_kP_{k+1}) < x$.

3.



Suppose that $\triangle A_1B_1C$ has area-measure 1. $\overline{CM_1}$ is the median of $\triangle A_1B_1C$ from C. For each n, A_{n+1} is the midpoint of $\overline{A_nC}$, and $\overline{A_{n+1}M_{n+1}} \parallel \overline{A_nM_n}$.

(a) $\forall_n \sum_{p=1}^{n} K(\triangle A_pM_pA_{p+1}) =$

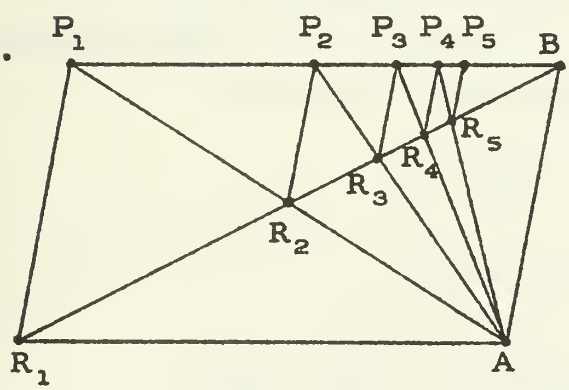
(b) $\forall_n \sum_{p=1}^{n} K(\text{⏢} A_pB_pB_{p+1}A_{p+1}) =$

(c) Find the smallest x such that

$$\forall_n \sum_{p=1}^{n} K(\text{⏢} A_pB_pB_{p+1}A_{p+1}) < x.$$

(d) Find the smallest x such that $\forall_n \sum_{p=1}^{n} K(\triangle A_pM_pA_{p+1}) < x.$

☆4.



<u>Given</u>: ABP_1R_1 is a parallelogram,
$R_1A = 1$,
$\forall_n\ \overline{R_{n+1}P_{n+1}} \parallel \overline{R_nP_n}$,
$\forall_n\ \{R_{n+1}\} = \overline{AP_n} \cap \overline{R_1B}$

<u>Find</u>: (a) $\forall_n \sum_{p=1}^{n} P_pP_{p+1} =$

(b) the smallest number x such that $\forall_n \sum_{p=1}^{n} P_pP_{p+1} < x$

INFINITE GEOMETRIC PROGRESSIONS

You learned in Unit 3 that the <u>repeating decimal</u>:

$$0.999\ldots \ [\text{or: } 0.\bar{9}\ ]$$

is a name for the number 1. Why should this be? To answer this, we begin by noting that, for example, the <u>terminating decimal</u> '0.568' is a name for

$$\frac{5}{10^1} + \frac{6}{10^2} + \frac{8}{10^3}.$$

So, perhaps the symbol '$0.\bar{9}$' is a name for

$$\frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots,$$

where the '...' means that we are supposed to keep on adding successive terms of the geometric progression "until they are all added up"! But, if this is so, '$0.\bar{9}$' is a name for the sum of <u>infinitely</u> many numbers, and what can 'the sum of infinitely many numbers' possibly mean? After all, addition is an operation on <u>pairs</u> of numbers. Thus, to find the sum of, say, 2, 3, and 7, you first add one of them to another, and then add the third to this sum. If there were a fourth number, you would add it to this last sum. So, as the recursive definition of Σ-notation shows, given any $n \in I^+$, the idea of the sum of the first n term of a sequence can be explained in terms of addition alone. But, something more is needed in an explanation of the notion of the sum of infinitely many terms.

In order to discover what this something more is, let's examine the terms of the sum sequence of the geometric progression

$$\frac{9}{10^1}, \frac{9}{10^2}, \frac{9}{10^3}, \ldots.$$

In particular, let's compare each sum with 1.

$$\frac{9}{10^1} = \frac{9}{10^1} \qquad 1 - \frac{9}{10^1} = \left(\frac{1}{10}\right)^1$$

$$\frac{9}{10^1} + \frac{9}{10^2} = \frac{99}{10^2} \qquad 1 - \frac{99}{10^2} = \left(\frac{1}{10}\right)^2$$

$$\frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} = \frac{999}{10^3} \qquad 1 - \frac{999}{10^3} = \left(\frac{1}{10}\right)^3$$

In general, since, for each n,

$$\sum_{p=1}^{n} \frac{9}{10^p} = \frac{\frac{9}{10}\left[1 - \left(\frac{1}{10}\right)^n\right]}{1 - \frac{1}{10}}$$

$$= 1 - \left(\frac{1}{10}\right)^n,$$

it follows that, for each n,

$$(*) \qquad 1 - \sum_{p=1}^{n} \frac{9}{10^p} = \left(\frac{1}{10}\right)^n.$$

Now, by Theorem 152a, $\left(\frac{1}{10}\right)^n$ is always positive. So, instead of (*), we could write:

$$\left| 1 - \sum_{p=1}^{n} \frac{9}{10^p} \right| = \left(\frac{1}{10}\right)^n$$

We know that, for large n, $\left(\frac{1}{10}\right)^n$ is small. In fact, given any positive number y, as small as you please,

$$\left(\frac{1}{10}\right)^n < y$$

if n is sufficiently large. More explicitly, by Theorem 165, n is sufficiently large if

$$n \geq \frac{1}{y\left(1 - \frac{1}{10}\right)}.$$

Combining what we have deduced from Theorems 152a and 165, we see that

$$\forall_{y > 0} \forall_{n \geq 10/(9y)} \left| 1 - \sum_{p=1}^{n} \frac{9}{10^p} \right| < y.$$

Less explicitly,

$$\forall_{y > 0} \exists_m \forall_{n \geq m} \left| 1 - \sum_{p=1}^{n} \frac{9}{10^p} \right| < y.$$

It is customary to abbreviate this last statement by:

$$\lim_{n \to \infty} \sum_{p=1}^{n} \frac{9}{10^p} = 1$$

[The phrase '$\lim_{n \to \infty}$' is read as 'the limit as n approaches infinity of'.] It is this that we mean when we say that the sum of all the terms of the infinite geometric progression

$$\frac{9}{10^1}, \frac{9}{10^2}, \frac{9}{10^3}, \ldots$$

is 1, or, more briefly, that

$$\sum_{p=1}^{\infty} \frac{9}{10^p} = 1.$$

[Read '$\sum_{p=1}^{\infty}$' as 'sigma, from p = 1 to infinity of'.] The repeating decimal '$0.\bar{9}$' is just an abbreviation for '$\sum_{p=1}^{\infty} \frac{9}{10^p}$'.

Consider, for another example, the GP

$$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots, (-\frac{1}{2})^{p-1}, \ldots,$$

whose first term is 1 and whose common ratio is $-\frac{1}{2}$. The first six terms in its continued sum sequence are

$$1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \text{ and } \frac{21}{32}.$$

[Check this.] What are the 7th, 8th, and 9th terms of the continued sum sequence? What is the 100th term? The 101st? Can you guess what

$\sum_{p=1}^{\infty} \left(-\frac{1}{2}\right)^{p-1}$ is?

Since, by Theorem 167, for $r \neq 1$,

$$\forall_n \; s_n = \frac{a_1 - a_1 r^n}{1 - r},$$

it follows that, for each n,

$$\sum_{p=1}^{n}\left(-\frac{1}{2}\right)^{p-1} = \frac{1 - 1\cdot\left(-\frac{1}{2}\right)^{n}}{1 - -\frac{1}{2}}$$

$$= \frac{1}{1 - -\frac{1}{2}} - \frac{\left(-\frac{1}{2}\right)^{n}}{1 - -\frac{1}{2}}$$

$$= \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^{n}.$$

So, for each n,

$$\frac{2}{3} - \sum_{p=1}^{n}(-\frac{1}{2})^{p-1} = \frac{2}{3}\left(-\frac{1}{2}\right)^{n},$$

and, since $\left|\left(-\frac{1}{2}\right)^{n}\right| = \left(\frac{1}{2}\right)^{n}$,

$$\left|\frac{2}{3} - \sum_{p=1}^{n}\left(-\frac{1}{2}\right)^{p-1}\right| = \frac{2}{3}\left(\frac{1}{2}\right)^{n}.$$

Since $0 < \frac{1}{2} < 1$, we can, as before, apply Theorem 165. For each positive number y [however small], for each n, $\left(\frac{1}{2}\right)^{n} < \frac{3}{2}y$ if $n \geq \frac{1}{\frac{3}{2}y\left(1 - \frac{1}{2}\right)}$. Hence, $\frac{2}{3}\left(\frac{1}{2}\right)^{n} < y$ if $n \geq \frac{4}{3y}$. Consequently,

$$\forall_{y>0}\ \exists_{m}\ \forall_{n\geq m}\left|\frac{2}{3} - \sum_{p=1}^{n}\left(-\frac{1}{2}\right)^{p-1}\right| < y.$$

In other words,

$$\lim_{n\to\infty}\sum_{p=1}^{n}\left(-\frac{1}{2}\right)^{p-1} = \frac{2}{3}$$

or, more briefly:

$$\sum_{p=1}^{\infty}\left(-\frac{1}{2}\right)^{p-1} = \frac{2}{3}$$

EXERCISES

A. Compute.

1. $\sum_{p=1}^{\infty} \frac{3}{10^p}$ 2. $\sum_{p=1}^{\infty} \left(\frac{1}{2}\right)^p$ 3. $\sum_{p=1}^{\infty} \frac{9}{100^p}$

B. For each of the following exercises draw a picture 4 inches long of the segment $\overline{0, 1}$ and plot at least the first six terms of the continued sum sequence of the given GP.

1. $a_1 = \frac{1}{2}$, $r = \frac{1}{2}$ 2. $a_1 = 1$, $r = -\frac{1}{2}$ 3. $a_1 = \frac{1}{4}$, $r = 1$

C. Consider the sequence a such that, for each n,

$$a_n = \frac{(2n + 1)(-1)^{n+1} - 1}{2n(n + 1)}.$$

1. Compute the first five terms of a.

2. Compute the first five terms of the continued sum sequence of a.

3. Guess what $\sum_{p=1}^{\infty} a_p$ is.

4. Compute the first five terms of the sequence b such that, for each n,

$$b_n = \frac{1 + (-1)^{n+1}}{2(n + 1)}.$$

5. Prove: $\forall_n \sum_{p=1}^{n} a_p = \frac{1 + (-1)^{n+1}}{2(n + 1)}$ [Hint. Use Theorem 130.]

☆6. Prove: $\sum_{p=1}^{\infty} a_p = 0$

[Hint. $\forall_n \left| 0 - \sum_{p=1}^{n} a_p \right| = \frac{1 + (-1)^{n+1}}{2(n + 1)} \le \frac{1}{n + 1}$ [Why?]]

* * *

You have seen that it is, in a sense, possible to add up all the terms of certain sequences. For example, the "sum" of all the terms of the GP whose first term is 1 and whose common ratio is $-\frac{1}{2}$ is $\frac{2}{3}$. In general, for any sequence a, and any number s,

$$\sum_{p=1}^{\infty} a_p = s$$

if and only if

$$\forall_{y>0} \exists_m \forall_{n \geq m} \left| s - \sum_{p=1}^{n} a_p \right| < y.$$

However, there are many sequences for which there does not exist such a number s. For example, consider the GP for which $a_1 = \frac{1}{4}$ and $r = 1$. For each n, $\sum_{p=1}^{n} a_p = \frac{n}{4}$ and, no matter what number s we test, $\left| s - \sum_{p=1}^{n} a_p \right|$ will not be small for large values of 'n'. In fact, if $n > 4s + 4$ then $\left| s - \sum_{p=1}^{n} a_p \right| > 1$. So, although $1 > 0$, there is no m such that, for each $n \geq m$, $\left| s - \sum_{p=1}^{n} a_p \right| < 1$. Clearly, a similar argument could be given to show that no GP for which $r \geq 1$ has a sum [in the sense in which we are using the word 'sum'].

For another example, consider the GP for which $a_1 = 1$ and $r = -1$. In this case, if n is odd then $\sum_{p=1}^{n} a_p = 1$ while, if n is even, $\sum_{p=1}^{n} a_p = 0$. Here, again, the terms of the continued sum sequence fail to cluster around any particular number, and [in our sense] this GP has no sum. Similarly, no GP for which $r \leq -1$ has a sum.

For other GPs--those for which $-1 < r < 1$--we have the theorem:

Theorem 168.

For any GP, a, with common ratio r such that $|r| < 1$,

$$\sum_{p=1}^{\infty} a_p = \frac{a_1}{1 - r}.$$

The proof is very similar to the proof, given previously [on pages 8-140 and 8-141], that

$$\sum_{p=1}^{\infty} \left(-\frac{1}{2}\right)^{p-1} = \frac{2}{3}. \qquad \left[\frac{2}{3} = \frac{1}{1 - -\frac{1}{2}}\right]$$

* * *

D. Compute, using Theorem 168.

1. $\sum_{p=1}^{\infty} \frac{7}{2^p}$

2. $\sum_{p=1}^{\infty} \left(-\frac{2}{3}\right)^{p-1}$

3. $\sum_{p=1}^{\infty} \frac{1}{10^p}$

Sample. $1.2\overline{46}$

Solution. $1.2\overline{46} = 1.2 + 0.0\overline{46}$

$$= 1.2 + \sum_{p=1}^{\infty} 0.046(0.01)^{p-1}$$

$$= 1.2 + \frac{0.046}{1 - 0.01} = 1.2 + \frac{0.046}{0.99} = \frac{12}{10} + \frac{46}{990}$$

$$= \frac{12 \cdot 99 + 46}{990} = \frac{1200 + (46 - 12)}{990} = \frac{1234}{990}$$

$$= \frac{617}{495}$$

4. $2.1\overline{45}$ 5. $0.\overline{07}$ 6. $0.\overline{236}$ 7. $\overline{2.36}$

8. $92.\overline{8}$ 9. $0.\overline{142857}$ 10. $9.00\overline{35}$ 11. $10.00\overline{35}$

E. Recall from Unit 1 that, for each $x \geq 0$, $^+|x| = x$ and, for each $x \leq 0$, $^+|x| = -x$. Using the convention according to which numerals for numbers of arithmetic may be used to stand for the corresponding nonnegative real numbers [so that, for example, '2' may (and usually does) have the meaning of '$^+2$', and '$|-3|$' of '$^+|-3|$'] we have the following definition:

$$\forall_{x \geq 0}\ |x| = x \text{ and } \forall_{x \leq 0}\ |x| = -x$$

Prove the following theorems. [Some are parts of Theorem 169.]

1. $\forall_x\ |x| \geq 0$

2. $\forall_x\ |-x| = |x|$

3. $\forall_x \forall_y\ |x| \cdot |y| = |xy|$

   [Hint. First prove: $\forall_x \forall_y\ (|x| \cdot |y| = xy \text{ or } |x| \cdot |y| = -(xy))$]

4. $\forall_x \forall_{y \neq 0}\ |x/y| = |x|/|y|$

5. $\forall_x \forall_n\ |x^n| = |x|^n$

6. $\forall_x \forall_y\ [|x| < y \Longrightarrow -y < x < y]$

   [Hint. Use Theorem 98c, and note that $\forall_x\ |x|^2 = |x^2|$.]

7. $\forall_x \forall_y\ [-y < x < y \Longrightarrow |x| < y]$

8. $\forall_x \forall_y\ |x + y| \leq |x| + |y|$ 9. $\forall_x \forall_y\ |x| - |y| \leq |x + y|$

* * *

On the next page is a proof of Theorem 168 [note how the theorems in Part E are used].

Since, for $r \neq 1$, $\sum_{p=1}^{q} r^{p-1} = \frac{1 - r^q}{1 - r}$ [Why?], it follows that

$$\frac{1}{1-r} - \sum_{p=1}^{q} r^{p-1} = \frac{r^q}{1-r}.$$

Since, for $r < 1$, $r \neq 1$, and since $|r^q| = |r|^q$, it follows that, for $r < 1$,

$$(*) \qquad \left| \frac{1}{1-r} - \sum_{p=1}^{q} r^{p-1} \right| = \frac{|r|^q}{1-r}.$$

Since, for $|r| < 1$, $1 - r > 0$, it follows from Theorem 165 that, for $0 < |r| < 1$,

$$(**) \qquad \forall_{y > 0} \forall_n \left[n \geq \frac{1}{y(1-r)(1-|r|)} \Rightarrow |r|^n < y(1-r)\right]$$

Since, obviously, $\forall_{y>0} \forall_n\ |0|^n < y(1-0)$, (**) also holds for $|r| = 0$.

Combining (*) and (**), it follows that, for $|r| < 1$,

$$\forall_{y > 0} \exists_m \forall_{n \geq m} \left| \frac{1}{1-r} - \sum_{p=1}^{n} r^{p-1} \right| < y.$$

Now, for $a_1 \neq 0$, $|a_1| > 0$ and, for $y > 0$, $\frac{y}{|a_1|} > 0$. So, for $a_1 \neq 0$ and $|r| < 1$,

$$\forall_{y > 0} \exists_m \forall_{n \geq m} \left| \frac{1}{1-r} - \sum_{p=1}^{n} r^{p-1} \right| < \frac{y}{|a_1|}$$

--that is,

$$\forall_{y > 0} \exists_m \forall_{n \geq m} \left| \frac{a_1}{1-r} - \sum_{p=1}^{n} a_1 r^{p-1} \right| < y \text{ [Explain.]}.$$

Hence, for any GP with common ratio r such that $|r| < 1$,

$$\sum_{p=1}^{\infty} a_p = \frac{a_1}{1-r}.$$

[In particular, note that, for each x such that $|x| < 1$, $\sum_{p=1}^{\infty} x^{p-1} = \frac{1}{1-x}$.]

* * *

F. 1. Find the sums of these infinite geometric progressions.

(a) $12,\ 4,\ \frac{4}{3},\ \ldots$ (b) $5,\ 0.5,\ 0.05,\ \ldots$

(c) $\frac{7}{2},\ \frac{7}{10},\ \frac{7}{50},\ \ldots$ (d) $\sqrt{14},\ \sqrt{2},\ \frac{\sqrt{14}}{7},\ \ldots$

2. A ball is dropped from a height of 50 feet. Each time it hits the ground it bounces to a height 80% of that from which it fell. Find the distance that the ball has moved by the time it hits the ground the fifth time. What distance will it have moved by the time it stops?

3. An equilateral triangle is inscribed in a circle of radius 16. A circle is then inscribed in the triangle, and a second such triangle is inscribed in the second circle. A third circle is then inscribed in the second triangle, etc. Find the sum of the area-measures of all such triangles.

4. If the sum of an infinite geometric progression is 18 and its first term is 3, what is the common ratio?

5. If the sum of the first two terms of a GP is nine times the sum of the next two terms and the sum of all the terms is 3, what is the first term?

6. The sum of all the terms of a geometric progression is 2, and so is the sum of the squares of these terms. What is the sum of the cubes of these terms?

7. The sum of the first two terms of a geometric progression is 2 and each term is twice the sum of all the terms that follow. Find the first term and the common ratio.

☆G. Find all solutions of $\text{`}\frac{EVE}{DID}\text{'} = .\overline{TALK}$, when different letters stand in place of different digits,

(a) if EVE and DID are relatively prime, and

(b) if EVE and DID are not relatively prime.

☆BASE-m APPROXIMATIONS OF POSITIVE REAL NUMBERS

As you know, each positive integer has *a decimal* [or base-ten] *representation*. If you studied pages 8-106 through 8-107, you also know that, for any integer $m > 1$, each positive integer has *a base-m representation*. For example, since

$$3805 = 1 \cdot 6^0 + 4 \cdot 6^1 + 3 \cdot 6^2 + 5 \cdot 6^3 + 2 \cdot 6^4,$$

the base-6 representation of 3805 is '25341'. [If you studied pages 7-107 through 7-111 of Unit 7, you learned an algorithm for finding in succession, the digits '1', '4', '3', '5', and '2' of this representation.]

You also know that some positive numbers which are not integers have terminating decimal representations. For examples, $\frac{1}{2} = 0.5$, $\frac{7}{4} = 1.75$, and $\frac{17491}{500} = 34.982$. And, finally, you know that some other positive numbers can be approximated, as closely as you wish, by numbers which have terminating decimal representations. For examples, $\frac{2}{11}$ is approximated by 0.1, by 0.18, by 0.181, by 0.1818, etc., and $\sqrt{2}$ is approximated by 1.4, 1.41, 1.414, 1.4142, etc. In each case, the qth approximation is in error by less than $1/10^q$.

It is easy to determine which positive numbers have terminating base-m representation. Do so. [*Hint*. Which positive numbers have terminating decimal representations?] Consideration of the division-with-remainder algorithm should convince you that each rational positive number has either a terminating or [like $\frac{2}{11}$], at worst a periodic decimal representation. And, from your work with "infinite geometric progressions", you know that each repeating decimal represents a rational number. Similar considerations show that a positive number has a periodic [or terminating] base-m representation if and only if it is rational.

We wish now to complete these considerations by proving that, for any integer $m > 1$, each positive *real* number has a base-m representation, "possibly a nonterminating one". To see what is meant by this, let's recall precisely what is meant by saying that each positive integer has a base-m representation. This means that, for any n, there is a positive integer p and there are nonnegative integers n_k, for $0 < k < p$, which are less than m and are such that

$$(*) \qquad n = \sum_{k=0}^{p-1} n_k m^k .$$

When we say that each positive number has a possibly nonterminating base-m representation, we mean that, for any $c > 0$, there is a positive integer p and there are nonnegative integers n_k, for all $k < p$, which are less than m and are such that, for each positive integer q,

$$(**) \qquad 0 \leq c - \sum_{k=-q}^{p-1} n_k m^k < \frac{1}{m^q}.$$

[For $m = 10$ and $c = \sqrt{2}$, $p = 1$ and $n_0 = 1$, $n_{-1} = 4$, $n_{-2} = 1$, $n_{-3} = 4$, $n_{-4} = 2$, etc. For $m = 10$ and $c = 2/11$, $p = 1$ and $n_0 = 0$, $n_{-1} = 1$, $n_{-2} = 8$, $n_{-3} = 1$, $n_{-4} = 8$, etc. For $m = 10$ and $c = \frac{17491}{500}$, $p = 2$ and $n_1 = 3$, $n_0 = 4$, $n_{-1} = 9$, $n_{-2} = 8$, $n_{-3}\ 2$, $n_{-4} = 0$, etc.] A consequence of (**) and Theorem 165 is that

$$c = \lim_{q \to \infty} \sum_{k=-q}^{p-1} n_k m^k.$$

This is sometimes abbreviated to:

$$c = \sum_{k=-\infty}^{p-1} n_k m^k$$

To see how to establish (**) it will be helpful to review the proof given on pages 8-106 and 8-107 for (*). And, before doing this, it will be helpful to consider, as an example, the equation:

$$3805 = 1 \cdot 6^0 + 4 \cdot 6^1 + 3 \cdot 6^2 + 5 \cdot 6^3 + 2 \cdot 6^4$$

From this equation we can see two ways of obtaining, for example, the digit '3':

$$\left[\!\!\left[\frac{3805}{6^2}\right]\!\!\right] = 3 + 5 \cdot 6 + 2 \cdot 6^2$$

$$\left[\!\!\left[\frac{3805}{6^3}\right]\!\!\right] = 5 + 2 \cdot 6$$

$$\left[\!\!\left[\frac{3805}{6^2}\right]\!\!\right] - \left[\!\!\left[\frac{3805}{6^3}\right]\!\!\right] 6 = 3$$

$$\left[\!\!\left[\frac{3805}{6^2}\right]\!\!\right] = 3 + 5 \cdot 6 + 2 \cdot 6^2$$

$$\left[\!\!\left[\frac{3805}{6^2}\right]\!\!\right] / 6 = \frac{3}{6} + 5 + 2 \cdot 6$$

$$\left\{\!\!\left\{\frac{[\![3805/6^2]\!]}{6}\right\}\!\!\right\} = \frac{3}{6}$$

$$\left\{\!\!\left\{\frac{[\![3805/6^2]\!]}{6}\right\}\!\!\right\} 6 = 3$$

[The second method is that on which the algorithm illustrated on page 7-110 of Unit 7 is based.] Either method can be used to find the digits in the base-m representation of any positive integer n. That they both give the same result is the main content of Theorem 124.

The proof [see page 8-106] that, for any $m > 1$, each positive integer n has a base-m representation depends mainly on Theorems 124 and 138. Briefly, for each $k \geq 0$,

$$\left[\!\left[\frac{n}{m^k}\right]\!\right] - \left[\!\left[\frac{n}{m^{k+1}}\right]\!\right] m$$

is an integer and, by Theorem 124, is nonnegative and less than m [and so, is a number represented by a base-m digit]. Next, by [a slight modification of] Theorem 138, for each p,

$$\sum_{k=0}^{p-1}\left(\left[\!\left[\frac{n}{m^k}\right]\!\right] - \left[\!\left[\frac{n}{m^{k+1}}\right]\!\right] m\right) m^k$$

$$= \sum_{k=0}^{p-1}\left(\left[\!\left[\frac{n}{m^k}\right]\!\right] m^k - \left[\!\left[\frac{n}{m^{k+1}}\right]\!\right] m^{k+1}\right)$$

$$= \left[\!\left[\frac{n}{m^0}\right]\!\right] m^0 - \left[\!\left[\frac{n}{m^p}\right]\!\right] m^p$$

$$= n - \left[\!\left[\frac{n}{m^p}\right]\!\right] m^p .$$

Now, for each p such that $n < m^p$, $\left[\!\left[\frac{n}{m^p}\right]\!\right] = 0$. So, for such integers p,

$$n = \sum_{k=0}^{p-1}\left(\left[\!\left[\frac{n}{m^k}\right]\!\right] - \left[\!\left[\frac{n}{m^{k+1}}\right]\!\right] m\right) m^k$$

$$= \sum_{k=0}^{p-1} \left\{\frac{[\![n/m^k]\!]}{m}\right\} m \cdot m^k .$$

[Finally, if p is the <u>least</u> k such that $n < m^k$ then, since $m^{p-1} \leq n < m^p$,

it follows that $1 \leq n/m^{p-1} < m$, so that $1 \leq [\![n/m^{p-1}]\!] < m$ and

$$\frac{1}{m} \leq \frac{[\![n/m^{p-1}]\!]}{m} < 1.$$

Consequently, $\left\{\!\!\left\{\frac{[\![n/m^{p-1}]\!]}{m}\right\}\!\!\right\} \neq 0.$]

The result is that there is an integer p, and there are nonnegative integers $n_0, n_1, ---, n_{p-1}$ less than m, such that [$n_{p-1} \neq 0$ and]

$$n = \sum_{k=0}^{p-1} n_k m^k$$

--that is, each positive integer n has a base-m representation.

Now, the preceding proof can easily be modified to show that each positive real number has a, possibly nonterminating, base-m representation. Let's do this. In the first place, the only reason for restricting k to be nonnegative was that, when the proof was first given on page 8-106, no meaning had yet been given to negative integral exponents. So, generalizing the first step in the preceding analysis of the proof, we can say that, for any real number c, and any integer k,

$$\left[\!\!\left[\frac{c}{m^k}\right]\!\!\right] - \left[\!\!\left[\frac{c}{m^{k+1}}\right]\!\!\right]m$$

is an integer which, by Theorem 124, is nonnegative and less than m. Next, it follows [again by a modification of Theorem 138] that, for any positive integers p and q,

$$\sum_{k=-q}^{p-1}\left(\left[\!\!\left[\frac{c}{m^k}\right]\!\!\right] - \left[\!\!\left[\frac{c}{m^{k+1}}\right]\!\!\right]m\right)m^k$$

$$= \sum_{k=-q}^{p-1}\left(\left[\!\!\left[\frac{c}{m^k}\right]\!\!\right]m^k - \left[\!\!\left[\frac{c}{m^{k+1}}\right]\!\!\right]m^{k+1}\right)$$

$$= \frac{[\![cm^q]\!]}{m^q} - \left[\!\!\left[\frac{c}{m^p}\right]\!\!\right]m^p$$

$$= \left(c - \frac{\{\!\{cm^q\}\!\}}{m^q}\right) - \left[\!\!\left[\frac{c}{m^p}\right]\!\!\right]m^p.$$

Now, for c > 0, and for each p such that $c < m^p$, $\left[\!\left[\frac{c}{m^p}\right]\!\right] = 0$. So, for each such positive integer p,

$$c = \sum_{k=-q}^{p-1}\left(\left[\!\left[\frac{c}{m^k}\right]\!\right] - \left[\!\left[\frac{c}{m^{k+1}}\right]\!\right]m\right)m^k + \frac{\{\!\{cm^q\}\!\}}{m^q}$$

$$= \sum_{k=-q}^{p-1}\left\{\!\!\left\{\frac{[\![c/m^k]\!]}{m}\right\}\!\!\right\}m \cdot m^k + \frac{\{\!\{cm^q\}\!\}}{m^q} \qquad \text{[by Theorem 124].}$$

[By Theorem 151b and the cofinality principle, there is a positive integer k such that $c < m^k$. We shall choose p to be the least such positive integer. If c > 1 it then follows as in the earlier proof that $\left\{\!\!\left\{\frac{[\![c/m^{p-1}]\!]}{m}\right\}\!\!\right\}m \neq 0$. If 0 < c < 1 then p = 1 and, since $m^{p-1} \nleq c$, $\left\{\!\!\left\{\frac{[\![c/m^{p-1}]\!]}{m}\right\}\!\!\right\}m = 0$. For example, the terminating decimal representation of $\frac{1}{25}$ is '$\underline{0}$.04'.]

Since $0 \leq \{\!\{cm^q\}\!\} < 1$, it follows that, for each positive integer q,

$$0 \leq \frac{\{\!\{cm^q\}\!\}}{m^q} < \frac{1}{m^q}.$$

Hence,

$$0 \leq c - \sum_{k=-q}^{p-1}\left\{\!\!\left\{\frac{[\![c/m^k]\!]}{m}\right\}\!\!\right\}m \cdot m^k < \frac{1}{m^q}.$$

Since, for each k, $\left\{\!\!\left\{\frac{[\![c/m^k]\!]}{m}\right\}\!\!\right\}m$ is a nonnegative integer less than m, c has a, possibly nonterminating, base-m representation.

[Recalling the meaning of 'the approximation to c correct to q decimal places' you can see that

$$\sum_{k=-q}^{p-1}\left\{\!\!\left\{\frac{[\![c/m^k]\!]}{m}\right\}\!\!\right\}m \cdot m^k$$

--which, by the preceding computation is $[\![cm^q]\!]/m^q$--is "the approximation to c correct to q m-mal places."]

MISCELLANEOUS EXERCISES

1. Simplify.

   (a) $\sqrt{x^2 + 10x + 25}$ (b) $\sqrt{121 - 22m^2 + m^4}$ (c) $\sqrt{81x^{10}}$

2. If the ratio of A to B is $\frac{3}{4}$, what is the ratio of $2A - B$ to $3A - 2B$?

3. Given the formula: $Q = S_1 \frac{V - V_1}{4\pi}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)$, find a formula for V_1.

4. If the perimeter of a rectangle is 302 and the area-measure is 5460, what is the radius of the circumcircle?

5. Expand.

   (a) $(9x^3 + 2x^2 + 4x + 6)^2$ (b) $(x^6 + 3x^3 + 2x^2 + 1)(x^2 + 5x + 4)$

6. Prove that the product of two numbers is a quarter of the square of their sum less a quarter of the square of their difference.

7. True-or-false?

   (a) $\forall_{p > 2}$ [p is a prime number $\Longrightarrow \exists_n\ p = 2n + 1$]

   (b) $\forall_{p > 2}$ [p is a prime number $\Longleftrightarrow \exists_n\ p = 2n + 1$]

8. Suppose that quadrilateral ABCD is a square and that E is a point of $\overrightarrow{AB}$ such that $B \in \overline{AE}$. If $AB = 16$ and $BE = 8$, what are the area-measures of the regions into which $\overleftrightarrow{DE}$ cuts the region bounded by ABCD?

9. For what value of 'x' are $9 - 2x$, $4 + 3x$, and $7 - 5x$ consecutive terms of an arithmetic progression?

10. Consider the sequences a and b such that, for each n, $a_n = 2^n$ and $b_n = n!$ Find an integer m such that, for each $p > m$, $a_p < b_p$.

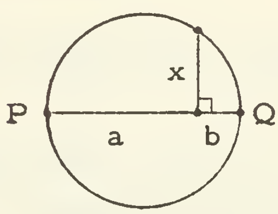
11. Write a quadratic equation whose roots are 0 and $-\frac{3}{2}$.

12. What is the reciprocal of $\left(\frac{8}{3}\right)^{-1}$?

13. Suppose that it takes 20 hours to assemble 24 mixing machines. If the assembling rate is increased by 25%, how long will it take to assemble 1 mixer?

14. Prove: $\forall_n \prod_{p=1}^{n} (2p - 1) = \frac{(2n)!}{2^n \cdot n!}$

15. Suppose that A, B, C, and D are points on a circle with center O such that ΔAOB and ΔCOD are similar. Prove that $\angle AOB \cong \angle COD$.

16. 

If $\overleftrightarrow{PQ}$ is a diameter of the circle and $PQ = a + b$ then $x =$

17. Consider a sequence a. Suppose that a_1, a_2, and a_3 are consecutive terms of an arithmetic progression, and that a_2, a_3, and a_4 are consecutive terms of a geometric progression. If $a_1 + a_4 = 11$ and $a_2 + a_3 = 10$, find the four terms.

18. What is the sum of the roots of the equation '$2x^2 + 9824x - 7 = 0$'?

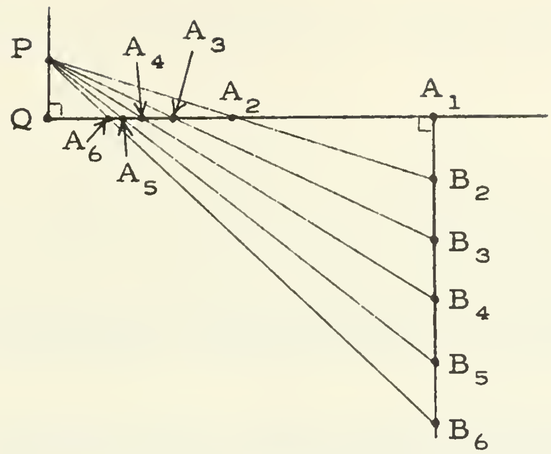
19. Will a strip of carpeting 3 feet wide and 11 feet 1 inch long fit in a 10-foot by 10-foot room?

20. If $f(x) = \frac{x^2 - 2x + 1}{3x^2 + x + 3}$ then $f(0) =$, $f(1) =$, $f(x^{-1}) =$, $f(x + 1) =$, and $f(y - 1) =$

21. Suppose that A is m yards north of B, C is n yards west of A, and D is n yards east of B. Prove that the distance between B and C is the distance between A and D.

22. Solve the equation: $\frac{2}{3}\left[3\left(\frac{15}{x} + \frac{1}{2}\right) - 8\right] - 1 = 5$

23.



<u>Given</u>: $\overleftrightarrow{PQ} \perp \overleftrightarrow{QA_1}$, $QA_1 = 1$, $\overleftrightarrow{QA_1} \perp \overrightarrow{A_1B_2}$, $PQ = A_1B_2 = B_2B_3 = B_3B_4 = \ldots$,

$\forall_n \{A_{n+1}\} = \overline{QA_1} \cap \overline{PB}_{n+1}$

<u>Find</u>: (a) $\forall_n \sum_{p=1}^{n} A_pA_{p+1} =$

(b) the smallest number x such that $\forall_n \sum_{p=1}^{n} A_pA_{p+1} < x$

24. If $2^{k+3} = 1024$ then $(k+3)^3 =$

25. Simplify.

(a) $\dfrac{2xy - 2y^2}{x^2 - y^2}$ (b) $\dfrac{x^4 - 1}{x^3 - 1}$ (c) $\dfrac{xy - y^2}{x^3 + x^2y - xy^2 - y^3}$

26. Consider the sequence a whose terms are sums of odd numbers.

$a_1 = 1$

$a_2 = 3 + 5$

$a_3 = 7 + 9 + 11$

$a_4 = 13 + 15 + 17 + 19$

$\cdots$

Guess a formula for the nth term and prove your guess correct.

27. Suppose that, for each $x \neq 1$, $f(x) = \dfrac{x}{x-1}$. Show that, for each $x \neq 1$, $f(f(x)) = x$.

28. Suppose that the perimeter of rectangle ABCD is 28 and its area-measure is 48. If $AB > BC$, what are the tangent-ratios for $\angle CAB$ and $\angle ACB$?

29. Suppose that 10 ordered pairs of a linear function are arranged in sequence such that the first components are consecutive terms of an arithmetic progression. Prove that the second components are also consecutive terms of an arithmetic progression.

30. A sequence of squares is constructed in the following manner. A side of the second square is a diagonal of the first, a side of the third square is a diagonal of the second, and so forth. If the perimeter of the first square is 1, what is the perimeter of the hundreth square?

31. Simplify.

(a) $y + 5 + \frac{6}{y - 2}$ (b) $\frac{-1}{3a} + \frac{7}{6a - 12} + \frac{3}{2(a - 2)}$

32. Prove that the sum of the squares of three consecutive odd numbers is 8 more than 3 times the square of the middle one.

33. Suppose that, for each $x \neq -1$, $f(x) = \frac{x - 1}{x + 1}$. Show that, for each x such that $-1 \neq x \neq 0$, $f(x^{-1}) = -f(x)$.

34. A wire 18 inches long is bent to form a circular sector. What should be the radius of the sector if its area is to be a maximum?

35. A man purchased a $2000 automobile on an installment plan. He paid $500 at the time of purchase, and the balance in semi-annual installments of $300 each together with 6% interest on all unpaid balances. How many dollars did he pay for the car?

36. The decimal numeral for 10! is '3628800' and its first nonzero digit counting from the right occurs in the 10^2 place. In what place does the first nonzero digit counting from the right occur in the decimal numeral for 10000!?

37. Show that there are no numbers x and y such that $x^2 - 16y^2 = 10$ and $x = 4y$.

38. Solve: $2^{6n+3} \cdot 4^{3n+6} = (8^n)^n$

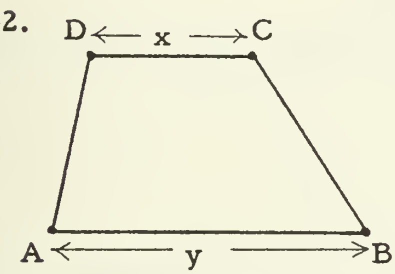
39. Simplify.

(a) $\dfrac{a^2}{a^3 - a^2b + ab^2 - b^3} \cdot \dfrac{a-b}{a^2+ab} \cdot \dfrac{a^4-b^4}{a-b}$ (b) $\left(\dfrac{x^2+y^2}{x} - y\right) \div \left(\dfrac{1}{y^3} + \dfrac{1}{x^3}\right)$

40. A spider is in the center of a 10-foot X 15-foot floor. If the ceiling is 7 feet high, how far will the spider walk if he takes the shortest path to one of the upper corners of the ceiling?

41. Suppose that, for each $x \neq 0$ and each k, $g(k) = x^k$. Show that, for each $x \neq 0$ and each k and j, $g(k)/g(j) = g(k-j)$.

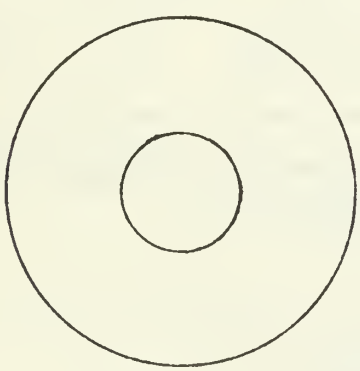
42.



Given: ABCD is a trapezoid with $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$

Locate the point $P \in \overline{AB}$ such that $\overleftrightarrow{CP}$ bisects the region bounded by ABCD.

43.



If the width of the circular ring is as long as a diameter of the inner circle, what per cent of the area-measure of the outer circle is the area-measure of the ring?

44. Prove: $\forall_x \; x^2 + x + 1 > 0$

45. Given n points on a circle such that no three chords having these points as end points are concurrent inside the circle. In how many points do the chords intersect inside the circle?

FACTORING

You have learned that

(1) $$\forall_x \forall_y\, x^2 - y^2 = (x - y)(x + y).$$

One use for this theorem is in factoring an expression which is "a difference of squares".

Example 1. Factor:

(a) $a^2 - b^2$

(b) $4u^2 - \frac{v^2}{9}$

(c) $(x - 1)^2 - (y + 3)^2$

(d) $a^4 - 16d^4$

Solutions:

$(a - b)(a + b)$

$\left(2u - \frac{v}{3}\right)\left(2u + \frac{v}{3}\right)$

$(x - y - 4)(x + y + 2)$

$(a^2 - 4d^2)(a^2 + 4d^2)$

In the case of Example 1(d) we can use (1) a second time to get:

$$(a - 2d)(a + 2d)(a^2 + 4d^2)$$

[We can go further if we allow ourselves to use radical signs. If we do this, another solution for Example 1(c) is:

$$(\sqrt{x - y} - 2)(\sqrt{x - y} + 2)(x + y + 2) \qquad [x \geq y]$$

Others are:

$$(\sqrt{x - 4} - \sqrt{y})(\sqrt{x - 4} + \sqrt{y})(x + y + 2) \qquad [x \geq 4 \text{ and } y \geq 0]$$

$$(x - y - 4)(\sqrt{x + 2} - \sqrt{-y})(\sqrt{x + 2} + \sqrt{-y}) \qquad [x \geq -2 \text{ and } y \leq 0]$$

In fact, if we allow the use of radical signs [and are satisfied to introduce restrictions], there is no end to the number of factors which we can pull out of a difference of squares (Explain.).]

EXERCISES

A. Factor by first transforming to a difference of squares.

1. $t^2 - 1$
2. $4a^2 - 9$
3. $1 - 16x^2$
4. $36x^2 - 49y^2$
5. $x^2 - \frac{y^2}{25}$
6. $16a^2 - \frac{9b^2}{25}$
7. $\frac{4}{x^2} - 1$
8. $\frac{8^2}{b^2} - \frac{1}{16c^2}$
9. $x^4 - y^2z^2$
10. $25a^4 - 1$
11. $x^2y^2 - 1$
12. $100a^2 - 4b^4c^2$

13. $\frac{a^4b^6}{4} - \frac{4}{c^2}$

14. $t^4 - \frac{s^4}{16}$

15. $\frac{1}{c^4} - \frac{25}{d^2}$

16. $\frac{x^8}{y^4} - \frac{1}{z^4}$

17. $(x + 5)^2 - (y - 2)^2$

18. $(2x - 3)^2 - (5y - 7)^2$

19. $(3a + 2b)^2 - (a - b)^2$

20. $(5t - 1)^2 - 1$

21. $(1 - 3s)^2 - 4$

22. $(7 - 2u)^2 - 9u^2$

23. $x^2 - 10x + 25 - y^2$

24. $a^2 - 2ab + b^2 - 1$

25. $4x^2 - 4xy + y^2 - 9a^2 - 6ab - b^2$

26. $y^2 + 4z^2 - 4u^2 - v^2 - 4yz + 4uv$

*

The theorem:

$$(1) \qquad \forall_x \forall_y\ x^2 - y^2 = (x - y)(x + y)$$

is closely related to a consequence of:

$$(2) \qquad \forall_{k \geq 0} \forall_z\ 1 - z^k = (1 - z)\sum_{p=1}^{k} z^{p-1}$$

[Compare with Theorem 153.]

The theorem (2) tells us, among other things, that

$$\forall_z\ 1 - z^2 = (1 - z)\sum_{p=1}^{2} z^{p-1}$$

or, in other words, that, for any c,

$$1 - c^2 = (1 - c)(1 + c).$$

So, for any $a \neq 0$ and any b,

$$1 - \left(\frac{b}{a}\right)^2 = \left(1 - \frac{b}{a}\right)\left(1 + \frac{b}{a}\right)$$

--that is,

$$a^2 - b^2 = (a - b)(a + b).$$

[Explain.]

Since it is easy to show that $0^2 - b^2 = (0 - b)(0 + b)$, the theorem (1) follows.

More generally, for any $k \geq 0$, any $a \neq 0$, and any b, it follows from (2) that

$$1 - \left(\frac{b}{a}\right)^k = \left(1 - \frac{b}{a}\right) \sum_{p=1}^{k} \left(\frac{b}{a}\right)^{p-1}.$$

So, since $a^k = a \cdot a^{k-1}$, it follows that

$$\left[1 - \left(\frac{b}{a}\right)^k\right] a^k = \left[\left(1 - \frac{b}{a}\right) a\right] \left[a^{k-1} \sum_{p=1}^{k} \left(\frac{b}{a}\right)^{p-1}\right].$$

Hence, using [among others] Theorems 159, 133, and 156,

$$a^k - b^k = (a - b) \sum_{p=1}^{k} a^{k-p} b^{p-1}.$$

Since, for $p < k$, $0^{k-p} = 0$, and since $0^0 = 1$, it follows that

$$0^k - b^k = (0 - b) \sum_{p=1}^{k} 0^{k-p} b^{p-1}. \qquad \text{[Explain.]}$$

Consequently, we have:

Theorem 170.

$$\forall_{k \geq 0} \forall_x \forall_y \; x^k - y^k = (x - y) \sum_{p=1}^{k} x^{k-p} y^{p-1}$$

Example 2. Factor: Solutions:

(a) $a^3 - b^3$ $\qquad (a - b)(a^2 + ab + b^2)$

(b) $u^6 - 64v^6$ $\qquad (u - 2v)(u^5 + 2u^4v + 4u^3v^2 + 8u^2v^3 + 16uv^4 + 32v^5)$

In the case of Example 2(b) we can apply Theorem 170 in another way:

$$\begin{aligned} u^6 - 64v^6 &= (u^3)^2 - (8v^3)^2 \\ &= (u^3 - 8v^3)(u^3 + 8v^3) \\ &= (u - 2v)(u^2 + 2uv + 4v^2)(u^3 + 8v^3) \end{aligned}$$

We have used here two instances of Theorem 170--one with $k = 2$, the

other with k = 3. We can use the same theorem a third time to factor '$u^3 + 8v^3$'. Here's how:

$$\begin{aligned} u^3 + 8v^3 &= u^3 - (-2v)^3 \\ &= [u - (-2v)][u^2 + u \cdot -2v + (-2v)^2] \\ &= (u + 2v)(u^2 - 2uv + 4v^2) \end{aligned}$$

So, finally, '$u^6 - 64v^6$' is equivalent to:

$$(u - 2v)(u + 2v)(u^2 + 2uv + 4v^2)(u^2 - 2uv + 4v^2)$$

*

B. Factor by transforming to a difference of like powers.

1. $x^3 - y^3$
2. $a^3 - 8b^3$
3. $1 - z^3$
4. $1 + z^3$
5. $1 + 27z^3$
6. $8k^3 - 27$
7. $x^5 - 1$
8. $y^5 + 32$
9. $32a^5 - b^5$
10. $x^6 - y^6$
11. $z^{12} - z^6$
12. $64a^3 - 27b^3$
13. $\frac{64}{x^3} - 1$
14. $\frac{10000}{x^4y^4} - \frac{z^4}{81}$
15. $\frac{1 + x^6}{x^3}$
16. $(x + 1)^4 - (y + 1)^4$
17. $25 - 16y^4$
18. $a^3 + b^3$
19. $a^{10} - z^5$
20. $x^4 - 64$
21. $a^2 + 2ab + b^2 - 81$

☆22. $a^3 - 3a^2b + 3ab^2 - b^3 - 1$

*

In factoring expressions it is often helpful to use other theorems in conjunction with Theorem 170. Among such theorems is:

$$\forall_x \forall_y \forall_z\ xy + xz = x(y + z)$$

This is, of course, the ℓdpma. It, together with the dpma and corresponding theorems about subtraction [Theorems 38 and 39], justify "taking out a common factor". When searching for factors of an expression, one should always begin by attempting to carry out this simplest kind of factoring. It is often possible to rewrite the given expression and then apply this procedure in such a way as to yield factors which can themselves be factored.

Example 3. Factor.

(a) $$\begin{aligned} yx^2 - 4y - x^2 + 4 &= (yx^2 - 4y) - (x^2 - 4) \\ &= y(x^2 - 4) - 1(x^2 - 4) \\ &= (y - 1)(x^2 - 4) \\ &= (y - 1)(x - 2)(x + 2) \end{aligned}$$

or:

$$\begin{aligned} yx^2 - 4y - x^2 + 4 &= (yx^2 - x^2) - (4y - 4) \\ &= x^2(y - 1) - 4(y - 1) \\ &= (x^2 - 4)(y - 1) \\ &= (x - 2)(x + 2)(y - 1) \end{aligned}$$

(b) $$\begin{aligned} 3x^2y^3 + 24x^5 &= 3x^2(y^3 + 8x^3) \\ &= 3x^2[y^3 - (-2x)^3] \\ &= 3x^2(y - -2x)[y^2 + y \cdot -2x + (-2x)^2] \\ &= 3x^2(y + 2x)(y^2 - 2xy + 4x^2) \end{aligned}$$

(c) $$\begin{aligned} 6t^2 - 9st + 6sr - 4rt &= (6t^2 - 9st) + (6sr - 4rt) \\ &= 3t(2t - 3s) + 2r(3s - 2t) \\ &= 3t(2t - 3s) - 2r(2t - 3s) \\ &= (3t - 2r)(2t - 3s) \end{aligned}$$

(d) $$\begin{aligned} a^3p^2 - 8b^3p^2 - 4a^3q^2 + 32b^3q^2 &= p^2(a^3 - 8b^3) - 4q^2(a^3 - 8b^3) \\ &= (p^2 - 4q^2)(a^3 - 8b^3) \\ &= (p - 2q)(p + 2q)(a - 2b)(a^2 + 2ab + 4b^2) \end{aligned}$$

(e) $$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^4 + 2x^2y^2 + y^4) - x^2y^2 \\ &= (x^2 + y^2)^2 - (xy)^2 \\ &= (x^2 + y^2 - xy)(x^2 + y^2 + xy) \\ &= (x^2 - xy + y^2)(x^2 + xy + y^2) \end{aligned}$$

(f) $$\begin{aligned} a^4 - 14a^2b^2 + 25b^4 &= (a^4 - 10a^2b^2 + 25b^4) - 4a^2b^2 \\ &= (a^2 - 5b^2)^2 - (2ab)^2 \\ &= (a^2 - 2ab - 5b^2)(a^2 + 2ab - 5b^2) \end{aligned}$$

Note how in Examples 3(e) and (f) the given expressions are transformed into differences of squares.

*

C. Factor.

1. $12x^3 - 30x^4y$
2. $8a^2b^3 - 24a^2b^2 + 8ab^3$
3. $m^2 + mp + mq + pq$
4. $t^2 - st + 5t - 5s$
5. $2z^4 - z^3 + 4z - 2$
6. $3px - qx - 3py + qy$
7. $y^2x^2 - 9y^2 - 6x^2 + 54$
8. $t^2k^2 - 16t^2 + 144 - 9k^2$
9. $9a^2b^3 - b^3 + 9a^2 - 1$
10. $x^8 + x^5 - x^3 - 1$
11. $4a^2 - 25b^2 + 2a + 5b$
12. $28a^5b + 64a^4b - 60a^3b$
13. $a^4 + 16a^2 + 256$
14. $x^4 - 6x^2y^2 + y^4$
15. $a^4 - 7a^2b^2 + b^4$
16. $x^4 + 4y^4$
17. $x^3 - 2x^2 + 4x - 8$
18. $x^4 - 5x^2 + 4$ [two methods]
19. $56 + 7u^3$
20. $t^3 + t^2w - tw^2 - w^3$

D. Prove these generalizations.

1. $\forall_n\ 3 | 2^n - (-1)^n$
2. $\forall_n\ \dfrac{2^n - (-1)^n}{3}$ is odd

☆3. $\forall_n\ 9 | 2^{2n} - 3n - 1$
☆4. $\forall_{m > 1}\ \forall_n\ (m^2 - 1)^2 | m^{2n} - (m^2 - 1)n - 1$

E. Solve these equations.

Sample. $m^2 - n^2 = 13$

Solution. $(m + n)(m - n) = 13$

Since 13 is a prime number, 13 and 1 are the only positive integral factors of 13. Hence, we know a solution must satisfy one of the sentences:

(a) $m + n = 13$ and $m - n = 1$

(b) $m + n = 1$ and $m - n = 13$

But, both m and $n \in I^+$, so we see that (b) has no solution.

From (a), we get: $2m = 14$ Hence, (7, 6) is the solution.

1. $m^2 - n^2 = 19$
2. $m^2 - n^2 = 60$
3. $m^3 - n^3 = 37$
4. $2^{2i} - 3^{2j} = 55$
5. $m^3 + n^3 = 133$
6. $m^3 - 2m^2 + 4m - 8 = 13$

*

☆7. Three couples played darts. At the end, each of the six people had an average score equal to the number of games he or she had played, and each boy had 63 points more than his date. John had played 23 more games than Mary had, and Henry had played 11 more than Kate. The other two people were Bill and Clara. Who was whose date?

F. Factor.

1. $(x + 3y)^2 - t^2$
2. $y^3 - y^2$
3. $b^2 - 12b - 85$
4. $2z^2 + 15z - 8$
5. $p^2q^2 - 81$
6. $9r^2 - (3r - 5s)^2$
7. $x^8 - 256^{-1}$
8. $1 - (a - b)^3$
9. $a^2 + 2ab + b^2 - c^2$
10. $x^2y^2 - 5xy - 24$
11. $125 + s^3$
12. $t^2 - a^2 + s^2 - 2ts$
13. $4a^4 + 9y^4 - 93a^2y^2$
14. $3a^2 + 5a + 2$
15. $10 - 77s + 15s^2$
16. $t^4 - r^2t^2 - 132r^4$
17. $1 - (x - y)^2$
18. $x^2 - 8ax + 16a^2 - 9b^2$
19. $a^4 + (2abc)^2 + 4b^4c^4$
20. $a^8 - 1$
21. $t^2 + 20t + 96$
22. $u^4 + v^4 - y^4 - x^4 + 2u^2v^2 - 2y^2x^2$
23. $2sx^2 + 3sxy - 2txy - 3ty^2$
24. $u^2 - a^2 + v^2 - b^2 - 2uv + 2ab$
25. $64 + m^3$
26. $1 - 27y^3$
27. $27m^3 + 1$
28. $(x + y)^2 - z^2$
29. $y^6 - 25$
30. $(5 - 2y)^2 - 16y^2$
31. $y^2 + b^2 - x^2 + 2by$
32. $(x + 2c)^2 - 1$
33. $(9x + 2)^2 - (8x - 3)^2$

G. Simplify.

1. $\dfrac{x^{1+n} + x^ny + xy^n + y^{1+n}}{x^n + y^n}$

2. $\dfrac{p^{p+1} + qp^p + rpq^{rq} + rq^{rq+1}}{p + q}$

3. $\dfrac{3x^2 + 10xy + 3y^2}{9x^2 + 6xy + y^2} \div \dfrac{x^2 + 3xy}{3xy + y^2}$

4. $\dfrac{a^2 + ab + ca + cb}{c + b} \times \dfrac{c^2 + cb}{a^2 + ab}$

MISCELLANEOUS EXERCISES

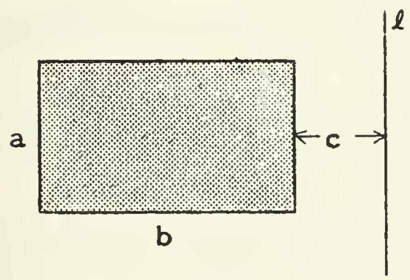
1. Simplify.

(a) $\dfrac{x^2 - x - 6}{x^2 + 4x + 4} \cdot \dfrac{x^2 - 2x - 8}{x^2 - 7x + 12}$ (b) $\left(\dfrac{a^2}{a^2 - b^2} - \dfrac{b^2}{a^2 + b^2}\right)\left(\dfrac{(a^2 - b^2)^2}{(a^2 - b^2)^2 + (a^2 + b^2)^2}\right)$

2. Simplify.

(a) $\dfrac{0.3}{0.02}$ (b) $\sqrt{1 - \left(\dfrac{\sqrt{3}}{2}\right)^2}$ (c) $\sqrt{20} - 2\sqrt{45}$ (d) $\dfrac{19}{11}\Big/\dfrac{19}{8}$

3. A lathe operator can do a certain job in 10 hours. If he increases his efficiency by 10%, how long will it take him to do the same job?

4. 

A rectangular ring is generated by revolving a rectangle 360° around a line ℓ parallel to one of its sides. If ℓ is at a distance c from a side with measure a, what is the volume-measure of the rectangular ring?

5. Suppose that rectangle ABCD and parallelogram ABEF have the same area-measure. Prove that the perimeter of the rectangle is not greater than that of the parallelogram.

6. Factor.

(a) $x^2 + 9x - 36$ (b) $a^2 - 4a - 117$ (c) $110 - x - x^2$

7. Study the following sentences and search for a pattern:

$$1 = 1$$
$$2 + 3 + 4 = 8 + 1$$
$$5 + 6 + 7 + 8 + 9 = 27 + 8$$
$$10 + 11 + 12 + 13 + 14 + 15 + 16 = 64 + 27$$

Guess a generalization and prove it correct.

8. Show that if $a/b = b/c$ then $a/c = (a + b)^2/(b + c)^2$.

COUNTING SUBSETS

In studying recursive definitions in Unit 7 you considered the number $C(j, k)[j \geq 0, k \geq 0]$ of k-membered subsets of a j-membered set, and used what you learned to solve a few counting problems [''How many ways can Milton walk to school?'', ''In how many ways can you distribute six pennies among four pockets?'', etc.] On page 7-72 you discovered a recursion formula:

$$(*) \qquad C(n, p) = C(n-1, p) + C(n-1, p-1)$$

which, together with the fact that $\forall_{j \geq 0}\ C(j, 0) = 1$ and $\forall_n\ C(0, n) = 0$, made it possible to compute values of the function C.

We will now make a further study of the function C and its application to the solution of counting problems. And, we shall begin by finding another recursion formula from which it is easy to derive an explicit definition for C.

Let us start with an example. Consider the 6-membered set S.

$$S = \{a, b, c, d, e, f\}$$

Our job, let us say, is to find a formula which will enable us to compute $C(6, 3+1)$ if we know $C(6, 3)$. Let $\mathcal{S}$ be the set of ordered pairs each of which has for second component a 4-membered subset of S, and for first component a 3-membered subset of its second component. Some of the ordered pairs in $\mathcal{S}$ are

$$\begin{array}{llll}
(\{a,b,c\}, \{a,b,c,d\}) & (\{a,b,c\}, \{a,b,c,e\}) & (\{b,c,d\}, \{b,c,d,e\}) & \ldots \\
(\{a,b,d\}, \{a,b,c,d\}) & (\{a,b,e\}, \{a,b,c,e\}) & (\{b,c,e\}, \{b,c,d,e\}) & \ldots \\
(\{a,c,d\}, \{a,b,c,d\}) & \vdots & \vdots & \\
(\{b,c,d\}, \{a,b,c,d\}) & & &
\end{array}$$

[Tell why $(\{b, c, f\}, \{c, d, e, f\})$ is not a member of $\mathcal{S}$.]

Since S has $C(6, 3+1)$ 4-membered subsets, there are $C(6, 3+1)$ categories [''columns''] of ordered pairs in $\mathcal{S}$ which have the same second component.

How many ordered pairs are there in each of these categories? There are as many as there are 3-membered subsets of a 4-membered set. And this number is $3 + 1$. [Explain.]

So, $\mathcal{S}$ has $(3+1) \cdot C(6, 3+1)$ ordered pairs.

Now let's count the ordered pairs of $\mathcal{S}$ from another point of view. This time we can classify them into categories which have the same first component. Here is a partial listing:

({a, b, c}, {a, b, c, d})	({a, b, d}, {a, b, c, d})	({c, e, f}, {a, c, e, f})	...
({a, b, c}, {a, b, c, e})	({a, b, d}, {a, b, d, e})	({c, e, f}, {b, c, e, f})	...
({a, b, c}, {a, b, c, f})	({a, b, d}, {a, b, e, f})	...	

[There is an error in this listing. Find it.]

Since S has C(6, 3) 3-membered subsets, there are C(6, 3) columns. How many ordered pairs are there in each column? There are as many as there are 4-membered subsets of S which contain a given 3 members of S. This number is 6-3 [Explain.]. So, S has (6-3)$\cdot$C(6, 3) ordered pairs.

Thus,

$$(3 + 1)\cdot C(6,\ 3 + 1) = (6-3)\cdot C(6,\ 3).$$

So,

$$C(6,\ 3 + 1) = C(6,\ 3)\cdot \frac{6 - 3}{3 + 1}.$$

Our main job, now, is to generalize this derivation and obtain a recursion formula which begins:

$$\forall_{j \geq 0} \forall_{k \geq 0}\ C(j,\ k + 1) = C(j,\ k)\cdot$$

[Can you guess what should be written after the multiplication dot?]

Consider, for some $j \geq 0$, a given j-membered set S. For a given $k \geq 0$, let $\mathcal{S}$ be the set of ordered pairs each of which has for second component a (k + 1)-membered subset of S, and for first component a k-membered subset of its second component. [If $k \geq j$ then, since $C(j,\ k+1) = 0$, $\mathcal{S} = \emptyset$.] Since S has C(j, k + 1) (k + 1)-membered subsets and since each of these has exactly k + 1 k-membered subsets [Why?], it follows that $\mathcal{S}$ has $(k + 1)\cdot C(j,\ k+1)$ members. On the other hand, if $k \leq j$ then each k-membered subset of S is a subset of exactly j - k (k + 1)-membered subsets of S [Explain.]. So, for $k \leq j$, $\mathcal{S}$ has $C(j,\ k)\cdot(j - k)$ members. Hence, at least for $k \leq j$,

$$(k + 1)\cdot C(j,\ k + 1) = C(j,\ k)\cdot(j - k).$$

Since, for $k > j$, $C(j,\ k+1) = 0 = C(j,\ k)$, this equation also holds for $k > j$. Consequently, for $j \geq 0$ and $k \geq 0$,

$$(**) \qquad C(j,\ k + 1) = C(j,\ k)\cdot \frac{j - k}{k + 1}.$$

From the recursion formula (**) and the fact that $\forall_j\ C(j, 0) = 1$ it is possible to compute any value of the function C with considerably less labor than is required when one uses (*), the formula of Unit 7. For example,

$$\begin{aligned}
C(12, 5) &= C(12, 4) \cdot \frac{12-4}{5} \\
&= C(12, 3) \cdot \frac{12-3}{4} \cdot \frac{8}{5} \\
&= C(12, 2) \cdot \frac{12-2}{3} \cdot \frac{9}{4} \cdot \frac{8}{5} \\
&= C(12, 1) \cdot \frac{12-1}{2} \cdot \frac{10}{3} \cdot \frac{9}{4} \cdot \frac{8}{5} \\
&= C(12, 0) \cdot \frac{12-0}{1} \cdot \frac{11}{2} \cdot \frac{10}{3} \cdot \frac{9}{4} \cdot \frac{8}{5} \\
&= 1 \cdot \frac{12}{1} \cdot \frac{11}{2} \cdot \frac{10}{3} \cdot \frac{9}{4} \cdot \frac{8}{5} \\
&= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 792
\end{aligned}$$

--a twelve-membered set has 792 5-membered subsets.

So, the function C may be defined recursively by:

$$\begin{cases} \forall_{j \geq 0}\ C(j, 0) = 1 \\ \forall_{j \geq 0}\ \forall_{k \geq 0}\ C(j, k+1) = C(j, k) \cdot \dfrac{j-k}{k+1} \end{cases}$$

EXERCISES

A. Compute.

1. C(5, 3)   2. C(9, 6)   3. C(10, 7)   4. C(3, 5)

B. Let $S = \{n: n \leq 10\}$.

1. What is the number of 7-membered subsets of S?
2. What is the number of 3-membered subsets of S?
3. How many 7-membered subsets of S do not contain the number 8?
4. How many 7-membered subsets of S contain the number 6?
5. How many 3-membered subsets of S contain only even numbers?

6. How many 3-membered subsets of S contain at least one even number and at least one odd number?

C. Prove.

1. $\forall_{j\geq 0}\, C(j, 1) = j$

2. $\forall_{j\geq 0}\, C(j, 2) = \frac{j(j-1)}{2}$

D. Prove by induction.

1. $\forall_n\, C(0, n) = 0$

2. $\forall_{j\geq 0}\forall_{k\geq j+1}\, C(j, k) = 0$

* * *

Recall that the factorial sequence is defined recursively by:

$$\begin{cases} 0! = 1 \\ \forall_{k\geq 0}\ (k+1)! = k!\cdot(k+1) \end{cases}$$

* * *

E. Compute.

1. $6!$

2. $1!$

3. $6!/4!$

4. $\frac{10!}{7!3!}$

5. $\frac{9!}{6!2!}$

6. $\frac{46!}{40!6!}$

F. Express using factorial notation.

1. $7\cdot 6\cdot 5$

2. $19\cdot 18\cdot 17\cdot 16$

3. $52\cdot 51\cdot 50\cdot 49\cdot 48$

* * *

The recursive definition for C on page 8-168 suggests an equivalent explicit definition:

Theorem 171.

$$\forall_{j\geq 0}\forall_{k\geq 0}\, C(j, k) = \frac{\prod_{i=0}^{k-1}(j-i)}{k!}$$

[For example, as previously shown,

$$C(12, 5) = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$= \frac{\prod_{i=0}^{4} (12 - i)}{5!} .]$$

In fact, Theorem 171 is easily derived by induction from the recursive definitions of C and of Π-notation. For part (i) of such a proof it is enough to note that

$$C(j, 0) = 1 = \frac{1}{1} = \frac{\prod_{i=0}^{-1} (j - i)}{0!} .$$

For part (ii), it follows from the recursive definition of Π-notation that, for $k \geq 0$,

$$\frac{\prod_{i=0}^{k-1} (j - i)}{k!} \cdot \frac{j - k}{k + 1} = \frac{\prod_{i=0}^{k} (j - i)}{(k + 1)!} . \qquad \text{[Explain.]}$$

Hence, by the recursive definition of C,

$$C(j, k) = \frac{\prod_{i=0}^{k-1} (j - i)}{k!} \Rightarrow C(j, k + 1) = \frac{\prod_{i=0}^{k} (j - i)}{(k + 1)!} .$$

Theorem 171 and the definition of the factorial sequence suggest a convenient way of expressing the values of C in terms of factorials, alone. For example,

$$C(12, 5) = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5!} = \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{5!} = \frac{7!(8 \cdot 9 \cdot 10 \cdot 11 \cdot 12)}{5!\,7!}$$

$$= \frac{12!}{5!\,7!} .$$

In fact, using Theorems 146, 148, 149, and 171, it is not hard to prove:

Theorem 172.

$$\forall_{j \geq 0} \forall_{k \geq 0}\ C(j+k,\ k) = \frac{(j+k)!}{j!\,k!}$$

For, by Theorem 171, it follows [for $j \geq 0$ and $k \geq 0$] that

$$C(j+k,\ k) = \frac{\prod_{i=0}^{k-1} (j+k-i)}{k!}.$$

By Theorem 149 ["reflection principle"], for $k \geq 0$,

$$\prod_{i=0}^{k-1} (j+k-i) = \prod_{i=0}^{k-1} [j+k-(k-1-i)]$$

$$= \prod_{i=0}^{k-1} [i+(j+1)]$$

$$= \prod_{i=j+1}^{j+k} i,$$

by Theorem 148. So, for $j \geq 0$ and $k \geq 0$,

$$C(j+k,\ k) = \frac{\prod_{i=j+1}^{j+k} i}{k!}$$

$$= \frac{\prod_{i=1}^{j} i \cdot \prod_{i=j+1}^{j+k} i}{\prod_{i=1}^{j} i \cdot k!}.$$

Now, by Theorem 146, for $j \geq 0$ and $j + k \geq j$, it follows that

$$\prod_{i=1}^{j+k} i = \prod_{i=1}^{j} i \cdot \prod_{i=j+1}^{j+k} i .$$

Hence [using factorial notation],

$$C(j + k, k) = \frac{(j + k)!}{j!\,k!} .$$

[Theorem 172 is sometimes restated as: For $0 \leq k \leq j$, $C(j, k) = \frac{j!}{k!\,(j - k)!}$]

As an immediate corollary of Theorem 172 one sees that, for $j \geq 0$ and $k \geq 0$,

$$C(j + k, k) = C(j + k, j)$$

[or that, for $0 \leq k \leq j$,

$$C(j, k) = C(j, j - k)$$

--a j-membered set has the same number of k-membered as of (j - k)-membered subsets].

* * *

G. Express in factorial notation.

1. $C(35, 32)$ 2. $C(176, 85)$ 3. $C(25, 5)$ 4. $C(25, 20)$

H. Compute, using either Theorem 171 or 172.

1. $C(10, 6)$ 2. $C(14, 10)$ 3. $C(25, 6)$ 4. $C(52, 5)$

5. $8 \cdot C(6, 2)$ 6. $C(9, 1) \cdot C(12, 7)$ 7. $C(45, 0)$ 8. $C(198, 1)$

☆I. Derive the theorem:

> Theorem 173.
>
> $$\forall_m \forall_n\ C(m, n) = C(m - 1, n) + C(m - 1, n - 1)$$

from Theorem 171 by using the recursive definition of Π-notation, and Theorems 147 and 148.

COMBINATIONS AND PERMUTATIONS

Many counting problems can be solved by using Theorem 171 [or Theorem 172] and the fact that the number of members of a cartesian product, $A \times B$, is the product of the numbers of members of its factors:

$$(*) \qquad N(A \times B) = N(A) \cdot N(B)$$

For example, in how many ways can one pick a committee of 5 from a group of 15 people? The number of such committees is, clearly, C(15, 5), and, by Theorem 171,

$$C(15, 5) = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 3003.$$

Suppose, now, that of the 15 people, 10 are seniors and 5 are juniors, and that the committee is to consist of 2 seniors and 3 juniors. In this case, each possible committee can be associated with an ordered pair whose first component is a set of 2 seniors and whose second component is a set of 3 juniors, and, by (*), the number of such ordered pairs is

$$C(10, 2) \cdot C(5, 3) = \frac{10 \cdot 9}{1 \cdot 2} \cdot \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 450.$$

Since, as remarked above, the possible committees can be matched in a one-to-one way with the ordered pairs, the number of possible committees is also 450.

In solving this problem we have used, besides (*) and Theorem 171, the if-part of an important principle about counting:

(C_1) $\left\{\right.$ Two sets have the same number of members if and only if the members of one set can be matched in a one-to-one way with those of the other.

As in the preceding problem, it is often convenient to replace a set whose members one wants to count [the set of committees] by another "equivalent" set [the set of ordered pairs of subcommittees] whose members are more easily counted.

Returning to the committee problem, let's complicate it further by supposing that the committee is to contain at least 1 senior and at least 2 juniors. Since it is to be a 5-person committee, it follows that it will consist of either 1 senior and 4 juniors, or 2 seniors and 3 juniors, or

3 seniors and 2 juniors. The numbers of committees of these three kinds are, respectively, $C(10, 1) \cdot C(5, 4)$, $C(10, 2) \cdot C(5, 3)$, and $C(10, 3) \cdot C(5, 2)$--that is [by Theorem 171], 50, 450, and 1200. Since each possible committee is of just one of these three kinds, the total number of possible committees is 50 + 450 + 1200--that is, 1700.

In solving this problem we have used another basic counting principle:

(C_2) $\left\{\right.$ If no two of a family of sets have a common member then the number of members in the union of the sets is the sum of the numbers of members in the individual sets.

How many of the possible 5-person committees contain at most 2 seniors? At least 3 seniors?

[It is interesting to notice that (*) on page 8-173 is a consequence of (C_1), (C_2), and the theorem:

$$\forall_{j \geq 0} \forall_{k \geq 0} \sum_{i=1}^{j} k = j \cdot k$$

[Compare this with (*) on page 8-32.]

For $A \times B$ is the union of $N(A)$ sets each of which has, by (C_1), $N(B)$ members, and no two of which have a common member [Explain.].]

A special case of (C_2) says that, for any sets A and B,

(**) $\quad A \cap B = \emptyset \Rightarrow N(A \cup B) = N(A) + N(B)$.

In solving some problems it is more convenient to use the fact that, for any [finite] sets A and B,

(***) $\quad N(A \cup B) = N(A) + N(B) - N(A \cap B)$.

One way to see that this is so is to notice that if you count the members of A, and then count those of B, you have counted each member of $A \cap B$ twice. [Although the proof is a little tricky, it is not difficult to derive (***) from (**) and the basic principles for sets given in Unit 5.] To see how (***) may be used, solve this problem:

Of 10 families, each of which has either an automobile or a television set, 8 have an automobile and 9 have a television set. How many have both? How many have an automobile but no television set?

EXERCISES

A. 1. There are 5 routes to and from the top of a mountain. In how many ways can a person go up and then down?

2. A man has 5 suits, 3 pairs of shoes, and 2 hats. In how many ways can he be dressed?

3. (a) In how many ways can one choose a committee of 5 from a group of 10 people?

   (b) In how many ways can one choose the 5-person committee if a certain 2 of the 10 people must belong to it? If a certain 2 of the 10 people are ineligible?

   (c) If half the people are men and half are women, in how many ways can the committee be chosen if it must contain 2 men and 3 women?

   (d) If half the people are men and half are women, in how many ways can the committee be chosen if it must contain an odd number of women? An even number of men? An even number of women?

4. A lot contains 50 articles, 6 of them defective.

   (a) How many selections of 5 articles can be made from the lot?

   (b) How many selections of 5 articles will contain at least 2 defective ones?

5. In how many ways can a committee of 7 men be chosen from a club with 25 members if 2 of the members refuse to serve together?

6. If 6 coins are tossed, in how many ways can it happen that 4 fall heads and 2 fall tails? That at most 2 fall tails?

7. In a certain examination, one is to choose 10 of 12 questions, including at least 1 of the first 2. In how many ways can he choose the 10 questions?

8. (a) In how many ways can 7 objects be divided into a group of 3 and a group of 4?

   (b) In how many ways can a group of 8 people divide into two groups of 4?

   (c) In how many ways can 8 people divide into two groups--one group of 4 to play bridge and the other canasta?

   (d) Show that the number of ways of dividing a group of $2n - 1$ objects into a group of $n - 1$ objects and one of n objects is the same as the number of ways of dividing a group of $2n$ objects into two groups of n objects.

9. Of a group of 39 black and white guinea pigs, 22 have some white and 29 have some black. How many are spotted? How many are pure white?

10. Generalize the formula:

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

by completing:

$$N(A \cup B \cup C) = N(A) +$$

[Hint. $A \cup B \cup C = (A \cup B) \cup C$]

11. In a high school class of 100, 95 take math, 85 take English, 73 take French, 70 take English and French, 80 take English and math, and 50 take math and French.

   (a) If every student in the class takes at least one of the three subjects, how many take all three?

   (b) If less than 40 students take all three subjects, what conclusion can you draw?

   (c) If it is reported that 48 students take all three subjects, what conclusion can you draw?

*

12. In how many ways can one choose two committees, one with 3 members and the other with 4 members, from a group of 15 people, if no one is to serve on both committees?

* * *

You may have solved Exercise 12 of Part A on the preceding page in any of several ways. One procedure is to say that there are C(15, 7) ways of choosing the total membership of the two committees and that, once this is done, there are C(7, 3) ways of dividing this into a 3-person committee and a 4-person committee. So, the total number of ways of choosing the two committees is C(15, 7)• C(7, 3). Another procedure is to say that there are C(15, 3) ways of choosing a 3-person committee and that, once this is done, there are C(12, 4) ways of choosing a 4-person committee from the remaining 12 people. So, the total number of ways of choosing the two committees is C(15, 3)• C(12, 4). [It is also C(15, 4)• C(11, 3). Explain.] In either case, of course, the same number is obtained:

$$\left.\begin{aligned} C(15, 7)\cdot C(7, 3) &= \frac{15!}{7!\,8!}\cdot\frac{7!}{3!\,4!} = \frac{15!}{8!\,3!\,4!} \\ C(15, 3)\cdot C(12, 4) &= \frac{15!}{3!\,12!}\cdot\frac{12!}{4!\,8!} = \frac{15!}{3!\,4!\,8!} \end{aligned}\right\} = 225225$$

For either solution, one uses a new counting principle:

$$(C_3)\left\{\begin{array}{l}\text{If a first event can occur in any of } m \text{ ways, and,}\\ \text{after it has occurred, a second event can occur in}\\ \text{any of } n \text{ ways, then the number of ways in which}\\ \text{the two events can occur successively is } mn.\end{array}\right.$$

This principle is much like (*) on page 8-173, and can be derived, using (C_1) and (C_2), in the same way as (*) was on page 8-174. In fact, (*) is the special case of (C_3) in which the ways in which the second event can occur are the same, no matter how the first event has occurred. For example, (*) can be used in solving Exercise 1 of Part A, since the mountaineer has the same 5 choices of ways to descend the mountain, no matter which way he has climbed it. But (C_3) would be useful in solving a similar exercise in which the mountaineer was required to return by a route different from the one by which he ascended. [(C_3) was also used in deriving the recursion formula (**) on page 8-167.]

* * *

B. 1. In how many ways can 10 people be sorted into 3 groups of 5, 3, and 2 people, respectively?

2. In how many ways can 6 people be split up into 3 pairs? [Hint. Recall Exercise 8(b) of Part A.]

3. In how many ways can 6 people be assigned to three jobs, each of which requires 2 people? [Hint. Recall Exercise 8(c) of Part A.]

4. A man lives within reach of 3 boys' schools, 4 girls' schools, and 2 coeducational schools. In how many ways can he send his 3 sons and 2 daughters to school?

5. An examination consists of three groups of questions--8 in the first group, 5 in the second, and 3 in the third. In how many ways can one choose 7 questions if he is to choose at least 3 from the first group and at least 1 from each of the other two groups?

6. A yacht club has 3 pennants of each of 5 colors, 15 pennants in all. How many different-looking arrangements of three pennants, one below another, can it hoist on its flag pole if

   (a) there is no restriction on the pennants used?

   (b) the three pennants must be of different colors?

   (c) no two adjacent pennants may have the same color?

7. Answer Exercise 6, assuming that the club has just 2 pennants of each color.

8. In how many ways can 6 people be seated in a row of 6 chairs? In a row of 8 chairs?

* * *

In many of the preceding exercises you have had to count the number of collections [or: combinations] of certain kinds. [Exercise 3(a) of Part A could be restated: What is the number of combinations of 10

things taken 5 at a time?] In these exercises it did not matter whether the members of a collection were arranged in any order. On the contrary, in Exercise 8 of Part B you had to count the number of ways in which 6 people could be arranged in order--say, from left to right. Similarly, Exercise 6(b) of Part B can be interpreted as asking for the number of ways in which one can choose 3 colors out of 5 and arrange them in order. In these and in similar exercises, one has to count, not merely collections, but ordered collections [or: permutations]. [The first question in Exercise 8 of Part B could be asked as follows: What is the number of permutations of 6 things? Exercise 6(b) is equivalent to: What is the number of permutations of 5 things taken 3 at a time?]

In answering the first question in Exercise 8 you might have proceeded as follows: Thinking of the chairs as designated as 1st chair, 2nd chair, 3rd chair, etc., there are 6 ways of choosing a person to occupy the 1st chair. Having made this choice, there are 5 ways of choosing a person to occupy the 2nd chair. So, by (C_3), there are $6 \cdot 5$ ways of choosing 2 people to occupy the first two chairs, respectively. Having made such a choice of 2 people, there are 4 ways of choosing a person to occupy the 3rd chair. So, by (C_3), there are $6 \cdot 5 \cdot 4$ ways of choosing 3 people to occupy the first three chairs, respectively. Continuing in this way, it is easily seen that there are $6!$ ways of seating 6 people in 6 chairs. [An alternative procedure is to designate the people as 1st person, 2nd person, 3rd person, etc., and, instead of choosing persons to occupy successive chairs, choose chairs to be occupied by successive persons.]

This suggests that the number $P(n)$ of permutations of a set of n things is $n!$. Acceptance of this involves agreeing that a set which has a single member can be ordered in just one way [the single member of such a set can be thought of as its 1st member]. With this agreement, we have part (i) of an inductive proof that $\forall_n\ P(n) = n!$: $P(1) = 1 = 1!$ For part (ii), suppose that each k-membered set can be ordered in $k!$ ways, and suppose that S is a $(k + 1)$-membered set. S can be ordered by choosing a 1st member in any of $k + 1$ ways and, having done so, choosing in any of $k!$ ways an ordering of the remaining k members. So, by (C_3), the number of ways of ordering S is $(k + 1) \cdot k!$--that is, $(k + 1)!$.

As a matter of fact, when one thinks more carefully of what it means to order a set, it becomes clear that the empty set can be ordered in just one way [it is ordered by the empty relation]. Since $0! = 1$, part (i) of the proof just given can be changed so that what we have actually proved is that

$$(*) \qquad \forall_{k \geq 0}\ P(k) = k!.$$

It is now easy to see how to compute the number $P(j, k)$ of permutations of j things taken k at a time. Such a permutation can be obtained by first choosing, in any of $C(j, k)$ ways, k of the things and, having done so, ordering them in any one of $P(k)$ ways. So, by (C_3),

$$\begin{aligned} P(j, k) &= C(j, k) \cdot P(k) \\ &= \frac{\prod_{i=0}^{k-1} (j - i)}{k!} \cdot k! \\ &= \prod_{i=0}^{k-1} (j - i). \end{aligned}$$

Consequently, we have:

<u>Theorem 174.</u>

$$\forall_{j \geq 0}\ \forall_{k \geq 0}\ P(j, k) = \prod_{i=0}^{k-1} (j - i)$$

[Alternatively, for $0 \leq k \leq j$, $P(j, k) = j!/(j - k)!$.] Notice that, for any $k \geq 0$, $P(k) = P(k, k) = k!$. So, theorem (*) for computing $P(k)$ is a corollary of Theorem 174.

Use Theorem 174 to answer the second question in Exercise 8 of Part B on page 8-178.

* * *

C. 1. In how many ways can a club with 21 members choose a president, vice-president, and secretary-treasurer, if no person can hold two of the three offices?

2. In how many ways can 4 mathematics books, 3 English books, and 5 French books be arranged on a shelf if

   (a) there is no restriction on their order?

   (b) books on the same subject are to be kept together?

3. In how many ways may 5 people be arranged around a circular table? [If each person moves one place to the right, the arrangement remains the same.]

4. In how many ways can 6 people be seated in a row of 6 seats if

   (a) 2 people insist on sitting next to each other?

   (b) 2 people refuse to sit next to each other?

5. How many 3-letter fraternity names can be formed using the 24 letters of the Greek alphabet if

   (a) no letter occurs twice in the same name?

   (b) there is no restriction on the number of occurrences of a letter?

6. How many line-ups are possible for a baseball nine if only 2 men can pitch, only 3 [other] men can catch, but each of the nine men can play in every other position?

7. In how many ways can n people be lined up in such a way that between some two given persons there are exactly p people?

8. (a) In how many ways can 5 keys be arranged on a ring?

   (b) In how many ways can n keys be arranged on a ring?

   (c) In how many ways can 2 keys be arranged on a ring?

9. In how many ways can 5 acts be arranged on a program so that 2 given acts occur in a specified order?

10. How many permutations are there of the letters in the word
(a) 'family'? (b) 'needed'?

* * *

You probably had no trouble with Exercise 10(a) of Part C, but Exercise 10(b) may have been more difficult. In both cases, the problem was to determine the number of permutations of 6 things, but in part (a) the things were of six kinds, while in part (b), 3 of the things were of one kind ['e's] and two were of another ['d's]. One reason that the answers to the two exercises are different is that interchanging, say, the 'i' and the 'y' in 'family' results in a different permutation of the letters, while interchanging the two 'd's in 'needed' does not.

Here is one way of seeing how to solve part (b):

Consider the "word":

$$ne_1e_2d_1e_3d_2$$

Just as for part (a), there are 6! permutations of the 6 "letters" in this "word". We can match these 720 permutations in pairs, putting two of them in the same pair if they differ only in the order of the 'd_1' and the 'd_2'. For example, we would match:

$$e_2, n, d_1, e_1, d_2, e_3 \quad \text{and:} \quad e_2, n, d_2, e_1, d_1, e_3$$

Each pair of permutations corresponds to just one permutation of the "letters" in the "word":

$$ne_1e_2de_3d$$

Consequently, the number of these latter permutations is $6!/2!$. Now, consider these 360 permutations. We can match these in sixes, putting 6 of them together if they differ only in the order of the 'e_1', 'e_2', and 'e_3' [Why 6?]. Each of these groupings corresponds to just one permutation of the letters in the word 'needed'. So, the number of such permutations is $(6!/2!)/3!$--that is, $\frac{6!}{2!\,3!}$.

In general, we have:

Theorem 175.

The number of permutations of p things, of which p_1 are of a first kind, p_2 of a second kind, ..., p_n are of an nth kind, and the remainder are of different kinds, is

$$\frac{p!}{\prod_{q=1}^{n} p_q!}.$$

* * *

D. 1. For each of the following words, tell the number of permutations of its letters.

(a) institution (b) letter (c) parallel (d) constitution

2. How many signals can be made by hoisting 5 pennants, 3 of which are red and 2 blue?

3. In how many ways can 4 men and 3 women be seated in 7 chairs if

(a) one considers them as 7 different people?

(b) one does not distinguish the men from one another, but does distinguish among the women?

(c) one does not distinguish among the men, nor among the women?

4. (a) In how many ways can 4 men and 3 women be seated alternately in a row of 7 chairs?

(b) In how many ways can this be done if one does not distinguish among the men?

(c) Repeat parts (a) and (b) for 4 men, 4 women, and 8 chairs.

(d) Repeat part (c), assuming that the chairs are arranged in a circle. [See Exercise 3 of Part C on page 8-181.]

5. Nine people are seated at a circular table and a bowl containing 4 apples, 3 oranges and 2 pears is passed. In how many ways can the fruit be distributed if each person takes one piece?

6. Solve Exercise 5 assuming that the basket contains at least 9 pieces of each kind of fruit.

7. In how many ways can 10 people distribute themselves in three rooms, 5 in the living room, 3 in the dining room, and 2 in the kitchen?

8. Another way to see how to solve Exercise 10(b) in Part C on page 8-182 is to think of it in this way: The problem is to pick out 3 of 6 positions in which to put 'e's and, then to pick out 2 of the remaining 3 positions in which to put 'd's. So, the number of permutations in question is $C(6, 3) \cdot C(3, 2)$. [Compare with the discussion, on page 8-177, of Exercise 12 of Part A.] Outline a proof of Theorem 175 based on this idea.

OTHER COMBINATORIAL PROBLEMS

In the preceding pages you have learned how to use the function C of Theorem 171, in combination with the counting principles (C_1) - (C_3), to solve a variety of counting problems. These uses of C were based on the fact that, for each $j \geq 0$ and $k \geq 0$, $C(j, k)$ is the number of k-membered subsets of a j-membered set [or, in other words, the number of combinations of j things taken k at a time].

In Unit 7 [specifically, in Exercise 1 on page 7-91] you learned two other counting theorems for subsets. The first of these had to do with the <u>total</u> number C_j of subsets of a j-membered set. If S is any $(j+1)$-membered set $[j \geq 0]$ and e_o is any member of S then $S = S_o \cup \{e_o\}$, where S_o is the j-membered subset of S which consists of the members of S other than e_o. Since each subset of S is either one of the C_j subsets of S_o or is obtained by adjoining e_o to such a subset, it follows that $C_{j+1} = C_j \cdot 2$. Since $C_o = 1$ [Explain.], it is easy to prove [using the recursive definition of the exponential sequence with base 2] that, for each $j \geq 0$, $C_j = 2^j$.

Theorem 176.

$$\forall_{j \geq 0}\ C_j = 2^j$$

Complete the proof of Theorem 176 begun above. [You will, later, discover another proof of this theorem based on the fact that, for each $j \geq 0$,

$$C_j = \sum_{k=0}^{j} C(j, k)\,.]$$

Some counting problems require one to find the total number of subsets of a given set, and, thus, can be solved by using Theorem 176. For example, on any given day all the members of your mathematics class may be present, or only some may be present, or all may be absent. How many situations of this kind can arise?

On the other hand, some problems require one to find the number of nonempty subsets of a given set. For example, how many weights can be measured on a balance if one has a set of standard weights consisting of a pound weight, a half-pound weight, a quarter-pound weight, and an eighth-pound weight? Assuming that the standard weights are to be used by putting some of them in one pan of the balance [with the thing to be weighed in the other pan] it is easy to see that the number of weights which can be measured is the number of nonempty subsets of a 4-membered set [Explain.].

[It is sometimes difficult to decide whether a problem calls for the number of subsets of a given set, or the number of its nonempty subsets. For example, how many sums of money can you pay out if you have just a penny, a nickel, a dime, and a quarter? In order to answer, you must decide whether, for a person who asked the question, "pay out" includes spending nothing.]

The other counting theorem which you discovered in Exercise 1 on page 7-91 of Unit 7 is

Theorem 177.

The number of odd-membered subsets of a nonempty set is the same as the number of its even-membered subsets.

Theorems 176 and 177 tell you that, for each n, an n-membered set has 2^{n-1} odd-membered subsets and 2^{n-1} even-membered subsets. One proof of Theorem 177 may be suggested to you by the proof, given on page 8-184, that, for each $j \geq 0$, $C_{j+1} = C_j \cdot 2$. [Later, you will discover another proof of Theorem 177.]

EXERCISES

1. Two roads connecting Zabranchburg and Griggsville are, themselves, connected by 8 crossroads. In how many ways can one go from one town to the other without retracing his route?

2. In how many ways can an odd-membered committee, with at least 3 members, be chosen from a club with 12 members?

3. What is the number of cartesian products whose first factor is a subset of a given m-membered set and whose second factor is a subset of a given n-membered set?

4. Suppose that $N(A) = m$, $N(B) = n$, and $A \cap B = \emptyset$. How many subsets of $A \cup B$ contain at least one member of each set?

* * *

☆There are a number of other types of counting problems which can be solved by using the function C. In order to discover some of these types, let's consider two problems which you already know how to solve:

(a) A student has 5 teachers and 2 apples. In how many ways can he distribute the apples among his teachers, giving at most one apple to each teacher?

(b) A school has 5 teachers and a student has 2 electives. In how many ways can he select his 2 courses, at most one from each teacher?

In each of these, the student is concerned with the number of 2-membered subsets of a 5-membered set--$C(5, 2)$. More generally, it is clear that

(1) the number of distributions of [or: ways of distributing] p things of the same kind in n boxes, at most one in each box, is $C(n, p)$

and

(2) the number of combinations of p things of n kinds, at most one of each kind, is $C(n, p)$.

Here is another pair of problems somewhat like (a) and (b):

(c) A boy has 3 pockets and 6 pennies. In how many ways can he distribute his pennies among his pockets?

(d) A bowl contains pears, apples, and oranges. In how many ways can one select 6 pieces of fruit from the bowl [assuming that it contains at least 6 pieces of fruit of each kind]?

Problem (c) asks for the number of ways of distributing p things of the same kind in n boxes [$p = 6$, $n = 3$], and problem (d) asks for the number of ways of selecting p things of n kinds [assuming that there are at least p things of each kind]. Evidently, the two problems have the same answer--the number of ways one can put p things into n boxes is [if one does not distinguish among the things] the same as the number of ways in which one can take p things out of n boxes [assuming that there are enough things in each box so that one doesn't run short]. Let's consider the problem of putting p things of the same kind into n boxes. If we think of the "boxes" as being formed by $n - 1$ vertical dividing walls in a row, from left to right, then a distribution of the p things can be thought of as putting them, also, in the same row. Those [if any] in the first box stretching out to the left of the first partition, those in the second box lined up between the first two partitions, etc. The result is a row of $n - 1 + p$ objects, $n - 1$ of which are partitions.

$$. \; . \; . \; | \; . \; | \; . \; . \qquad \leftarrow (3 - 1) + 6 \text{ objects}$$

A different distribution will, again, consist of a row of $n - 1 + p$ objects, a different $n - 1$ of which are partitions.

$$. \; | \; . \; . \; . \; . \; . \; | \qquad \leftarrow (3 - 1) + 6 \text{ objects}$$

So, the number of ways of distributing p things of the same kind in n boxes is the same as the number of ways of choosing n - 1 objects from n - 1 + p objects--that is, it is $C(n - 1 + p, n - 1)$ or, equivalently, $C(n - 1 + p, p)$.

Hence, in both problem (c) and problem (d), the number of ways is $C(3 - 1 + 6, 3 - 1)$--that is, 28. Generally,

> (3) the number of distributions of p things of the same kind in n boxes is $C(n - 1 + p, n - 1)$,

and

> (4) the number of combinations of p things of n kinds is $C(n - 1 + p, n - 1)$ [assuming that there are at least p things of each kind to select from].

Suppose, now, that we modify problems (c) and (d) by requiring, in (c), that the boy put at least one penny in each pocket and, in (d), that one select at least one piece of fruit of each kind. As in the case of the original problems (c) and (d) it is clear that the two new problems have the same answer. [Explain.] Let's consider the first one. The boy can distribute his pennies by first putting 1 penny in each pocket and then distributing the remaining 3 pennies. Since he is not distinguishing one penny from another, the number of distributions of his 6 pennies, at least one in each pocket, is just the number of ways he can distribute 3 pennies. By (3) this is $C(3 - 1 + 3, 3 - 1)$--that is, 10. In general, the number of ways of distributing p things among n boxes, at least 1 thing in each box is just the total number of ways in which p - n things can be distributed among n boxes. Since, by (3), the latter is $C(n - 1 + p - n, n - 1)$, we have that

> (5) the number of distributions of p things of the same kind in n boxes, at least 1 in each box, is $C(p - 1, n - 1)$,

and

> (6) the number of combinations of p things of n kinds, at least 1 of each kind, is $C(p - 1, n - 1)$ [assuming that there are at least $p - n + 1$ things of each kind to select from].

[It is easy to generalize (5) and (6) to take care of situations which can be described by replacing 'at least 1' by 'at least q'. Do so.]

Besides the parallelism between distributions of p things of the same kind in n boxes and combinations of p things of n kinds, which is illustrated by (1) and (2), (3) and (4), and (5) and (6), there is a parallelism between distributions of p things of different kinds in n boxes and permutations [or: arrangements] of p things of n kinds. To see how this is, let's, again, begin with two problems which are easy to solve:

(a′) Five people live in a 7-room house. In how many ways can they choose rooms to go to, if each wants to be alone?

(b′) A small boy has 7 cubical blocks, all of different colors. How many different looking walls can he build, each 5 blocks long?

The two problems, like problems (a) and (b), have the same answer: $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$ or, more simply: 2520. In general,

(1′) the number of distributions of p things of different kinds in n boxes at most 1 in each box, is $C(n, p) \cdot p!$,

and

(2′) the number of permutations of p things of n kinds, at most 1 of each kind, is $C(n, p) \cdot p!$ [See Theorem 174. For $n \geq p$, $C(n, p) \cdot p! = n!/(n-p)!$.]

Next, consider these two problems:

(c′) A boy has 3 pockets and 5 coins--a penny, a nickel, a dime, a quarter, and a half-dollar. In how many ways can he pocket these coins?

(d′) A bowl contains pears, apples, and oranges. In how many ways can one line up 5 pieces of fruit from the bowl [assuming that it contains at least 5 pieces of fruit of each kind]?

As usual, the two problems have the same answer--this time '3^5'. [You can explain this by using the counting principle (C_3). Do so.]

In general,

(3′) the number of distributions of p things of different kinds in n boxes is n^p,

and

(4′) the number of permutations of p things of n kinds is n^p [assuming that there are available at least p things of each kind.]

[Compare (3′) and Theorem 176.]

Finally, consider the problems:

(e′) There are 12 people at a party which is being held in 3 rooms. At a certain time there are 5 people in the first room, 3 in the second, and 4 in the third. In how many ways can this happen?

(f′) A small boy has 12 blocks of the same size, 5 red, 3 green, and 4 yellow. How many different looking walls can he build, each wall being 12 blocks long?

If you imagine the 12 people of problem (e′) as standing in a row, and of the small boy of problem (f′) as putting one of his blocks at the feet of each person, red blocks for people who were in the first room, green blocks for the second room, and yellow blocks for the third room, you will see that the two problems have the same answer. And, by Theorem 175, the answer is $\frac{12!}{5!3!4!}$, or, more simply: 27720 [Another way of arriving at the answer, from the point of view of problem (e′), is suggested by Exercise 8 on page 8-184.]

In general,

(5′) the number of distributions of p things of different kinds in n boxes, with, for $1 \le i \le n$, p_i things in the ith box, is

$$\frac{p!}{\prod_{i=1}^{n} p_i!},$$

and

(6′) the number of permutations of p things of n kinds, with, for $1 \le i \le n$, p_i things of the ith kind, is

$$\frac{p!}{\prod_{i=1}^{n} p_i!}.$$

In learning to solve counting problems it is helpful to discover as many ways as you can for analyzing each problem. Practice at this gives you a better chance of finding <u>one</u> way to solve a new problem. Here, as an example, is a problem which can be analyzed in several ways:

> In going from his home to his job, John has to walk 12 blocks--8 blocks east and 4 blocks north. How many 12-block routes can he take?

One way to analyze this problem is to note that at the beginning of each block, John has to choose whether to go east or to go north. So, he has 12 choices, 8 for east and 4 for north. The number of possible routes is just the number of ways of arranging his 12 choices, 8 of one kind and 4 of another. By (6′), this number is $\frac{12!}{8!4!}$--that is, 495.

A slightly different way of thinking of the problem leads to using (5′). Think of his 12 choices in order--his 1st choice, when he starts to walk, his 2nd choice at the beginning of the second block, etc. John has these 12 "things" of different kinds to put into 2 boxes--8 in the east-box and 4 in the north-box. By (5′), the number of ways he can do this is, again, $\frac{12!}{8!4!}$.

As a matter of fact, once he has decided which 8 of his 12 choices are to be east-choices, his route is fixed. So, the number of routes is just the number of ways to choose 8 things from 12. This is C(12, 8) or, again, $\frac{12!}{8!4!}$.

Finally, his route will be determined once he decides how many blocks he is going to walk east on each of the 5 available east-west streets [Why 5?]. So, the number of possible routes is just the number of distributions of 8 things of the same kind in 5 boxes. By (3), this is C(5 - 1 + 8, 5 - 1) or, again $\frac{12!}{4!8!}$.

This last analysis suggests another problem: For how many 12-block routes will John walk at least 1 block on each east-west street? By (5), the number of such routes is C(8 - 1, 5 - 1)--that is, 35.

For how many 12-block routes will John walk at least 1 block on each north-south street? At least 1 block on each north-south street <u>and</u> at least one block on each east-west street?

☆EXERCISES

1. In how many ways can 2 sixes, 3 fives, and an ace be thrown with 6 dice?

2. (a) In how many ways can 12 dice fall?

   (b) In how many ways can 12 dice fall so that each of the 6 possible faces appears twice [2 aces, 2 deuces, ..., and 2 sixes]?

3. (a) In how many ways can 52 cards be dealt to four bridge players?

   (b) In how many ways can the cards be dealt so that each player gets a single ace?

4. (a) What is the number of possible bridge hands?

   (b) How many of the possible bridge hands contain exactly 5 hearts?

   (c) How many contain exactly p red cards [$p \leq 13$]?

5. (a) In how many ways can 11 books be divided among 3 people, 2 of whom are to receive 4 books each?

   (b) In how many ways can 11 books be wrapped in 3 parcels, 2 of which are to contain 4 books each?

6. In how many ways can a dozen cans of soup be chosen from a shelf holding cans of each of 5 kinds of soup?

7. How many dominoes are there in a set containing one of each kind of domino from double blank to double nine?

8. A florist has 4 kinds of roses. In how many ways can he make up a bunch of 12 roses?

9. In how many ways can 10 oranges be distributed among 6 people?

10. Out of 16 consecutive positive integers, in how many ways can one choose 3 whose sum is even? Whose sum is odd?

11. A post office has 8 kinds of stamps. In how many ways can a person buy

    (a) 12 stamps? (b) 6 stamps? (c) 6 stamps of different kinds?

12. Call a set of positive integers <u>unfriendly</u> if no two of its members are consecutive. How many unfriendly p-membered sets are there, each of whose members is at most n?

☆A FOURTH COUNTING PRINCIPLE

In Exercise 10 of Part A on page 8-176 you used the fact that, for any finite sets A and B,

$$(*) \qquad N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

to show that, for any finite sets A, B, and C,

$$(**) \qquad N(A \cup B \cup C) = N(A) + N(B) + N(C) - [N(A \cap B) + N(B \cap C) + N(C \cap A)] + N(A \cap B \cap C).$$

These two results suggest the following counting principle:

(C_4) If a set A is the union of n subsets, $A_1, A_2, \ldots, A_n$ and if N_1 is the sum of the numbers of members of the subsets, N_2 is the sum of the numbers of members of the intersections of the subsets 2 at a time, N_3 is the sum of the numbers of members of the intersections of the subsets 3 at a time, etc., then

$$N(A) = \sum_{i=1}^{n} (-1)^{i-1} N_i .$$

For example, if $A = A_1 \cup A_2$ then

$$N(A) = \sum_{i=1}^{2} (-1)^{i-1} N_i = N_1 - N_2$$

$$= [N(A_1) + N(A_2)] - N(A_1 \cap A_2).$$

But, this is just another way of writing (*). Similarly, the case $n = 3$ is merely (**). Check this.

It is easy to justify (C_4) by using Theorem 177. Suppose that A is the union of $A_1, A_2, \ldots, A_n$, and that e is any member of A. Then, e belongs to one or more of the sets $A_1, A_2, \ldots, A_n$--say, it belongs to p of them. Since $p = C(p, 1)$, e is counted $C(p, 1)$ times in the sum N_1. Since e belongs to p of the sets, it belongs to $C(p, 2)$ of their 2-at-a-time intersections. So, it is counted $C(p, 2)$ times in the sum N_2. Similarly, for $2 < i \leq p$, e is counted $C(p, i)$ times in the sum N_i. Since e belongs to only p of the sets, it does not belong to any of their i-at-a-time intersections for $i > p$. So, for $i > p$, e is not counted in the sum N_i. Consequently, in the alternating sum

$$\sum_{i=1}^{n} (-1)^{i-1} N_i,$$

e is counted

$$\sum_{i=1}^{p} (-1)^{i-1} C(p, i)$$

times. What (C_4) says, then, is that [for $p \leq n$]

$$\sum_{i=1}^{p} (-1)^{i-1} C(p, i) = 1.$$

Now, C(p, i) is the number of i-membered subsets of a p-membered set.

$$\sum_{i=1}^{p} (-1)^{i-1} C(p, i)$$

is the sum of the numbers C(p, i), $1 \le i \le p$, for i an odd number, minus the sum of the numbers C(p, i), $1 \le i \le p$, for i an even number. [Explain.] According to Theorem 177, a p-membered set has the same number of odd-membered subsets as it has even-membered subsets. Remembering the empty set--an even-membered subset, we see that what Theorem 177 tells us is that

$$\sum_{i=1}^{p} (-1)^{i-1} C(p, i) - 1 = 0.$$

And, this is just what we needed to know in order to complete the proof of (C_4).

☆EXERCISES

1. In a town with 3050 families,

   900 families have annual incomes of less than \$2000 each,
   2000 have annual incomes of at least \$2000 but less than \$4000,
   900 have annual incomes of at least \$3000 but less than \$5000,
   50 have annual incomes of at least \$5000.

   How many have annual incomes of at least \$4000 but less than \$5000?

2. Suppose that 10 letters, each to a different person, are put at random into 10 envelopes addressed to these people. In how many of the 10! ways in which this can happen does at least one letter get into the proper envelope?

THE BINOMIAL THEOREM

Each of the following equations holds for all values of 'a' and 'b':

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

[Check the third of these equations by using the second: $(a + b)^3(a + b) = \ldots$] Guess a fourth equation, beginning:

$$(a + b)^5 = a^5 +$$

[Hint. Compare the right sides of the given equations with the rows in a table of values of 'C(n, k)'.]

One can expand, say, '$(a + b)^4$'--or, equivalently, '$(a + b)(a + b)(a + b)(a + b)$'--by using only the dpma and the ℓdpma:

$$\begin{aligned}
&(a + b)(a + b)(a + b)(a + b)\\
&= [a(a + b) + b(a + b)](a + b)(a + b)\\
&= [(aa + ab) + (ba + bb)](a + b)(a + b)\\
&= [(aa + ab)[a + b] + (ba + bb)[a + b]](a + b)\\
&= [(aa[a + b] + ab[a + b]) + (ba[a + b] + bb[a + b])](a + b)\\
&= [([aaa + aab] + [aba + abb]) + ([baa + bab] + [bba + bbb])](a + b)
\end{aligned}$$

etc.

The final line would begin:

$$= [([(aaaa + aaab) + (aaba + aabb)] + [(abaa + abab) + (abba + abbb)])$$
$$+ ([(baaa +$$

[How many 4-letter product-expressions would there be in the completed last line?]

Each of the 4-letter products corresponds with a choice of either the 'a' or the 'b' from each of the four '(a + b)' factors. Thus, there is one of them, 'aaaa', which corresponds with the choice of the 'a' from each factor; four, 'aaab', 'aaba', 'abaa', and 'baaa', which correspond with choices of the 'b' from one factor and the 'a' from each of the other three factors, etc. Using the apm, the cpm, and the definition of the exponential sequences, we see that there is one 4-letter product equivalent to 'a^4', four equivalent to 'a^3b', six to 'a^2b^2', four to 'ab^3', and one to 'b^4'. The first arises through choosing no 'b's out of the four which occur in the four factors, the next four arise through choices of one 'b' out of

the four, the next six through choices of two 'b's out of four, etc. Recalling that the number of choices of k things out of n is C(n, k), we see that, for each a and b,

$$(a+b)^4 = C(4, 0)a^4b^0 + C(4, 1)a^3b^1 + C(4, 2)a^2b^2 + C(4, 3)a^1b^3 + C(4, 4)a^0b^4$$
$$= \sum_{k=0}^{4} C(4, k)a^{4-k}b^k.$$

Complete: $(a + b)^n = \sum_{k=0}$

Later we shall give an inductive proof of <u>the</u> <u>binomial</u> <u>theorem</u>:

<u>Theorem 178.</u>

$$\forall_x \forall_y \forall_{j \geq 0}\ (x+y)^j = \sum_{k=0}^{j} C(j, k)x^{j-k}y^k$$

EXERCISES

<u>A</u>. Expand each <u>binomial</u> <u>exponential</u> by using the binomial theorem, and simplify.

<u>Sample</u>. $(x - y)^6$

<u>Solution</u>. $(x-y)^6 = (x+-y)^6 = \sum_{k=0}^{6} C(6, k)x^{6-k}(-y)^k$

$$= 1x^6 + \frac{6}{1}x^5(-y) + \frac{6\cdot 5}{1\cdot 2}x^4(-y)^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}x^3(-y)^3$$
$$+ \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}x^2(-y)^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}x(-y)^5$$
$$+ \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(-y)^6$$

$$= 1x^6 + 6x^5(-y) + 15x^4(-y)^2 + 20x^3(-y)^3$$
$$+ 15x^2(-y)^4 + 6x(-y)^5 + (-y)^6$$

$$= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$$

[In computing the <u>binomial</u> <u>coefficients</u> $C(6, k)$ we have used the fact that, for $k \geq 0$, $C(6, k+1) = \frac{6 - k}{k + 1} C(6, k)$ [recursive definition of C]. The seven expressions of the form '$C(6, k)x^{6-k}(-y)^k$' are called the <u>terms</u> of the binomial expansion of '$(x - y)^6$'. The one, for example, for which $k = 3$ is called the <u>fourth term</u>. The recursion formula for successive values of C justifies the following short cut:

> The binomial coefficient of the first term is 1, and that of each succeeding term is the product of the binomial coefficient of the immediately preceding term by the exponent of the first exponential factor in that term, divided by the number of that term [or: divided by 1 more than the exponent of the second exponential factor].

Using this rule one can write down the next-to-last step in the preceding solution very rapidly. Do this.]

1. $(a + b)^8$
2. $(x - y)^7$
3. $(2a + b)^4$
4. $(3x - 2y)^5$
5. $(7z - 1)^5$
6. $\left(\frac{x}{2} + \frac{y}{5}\right)^5$
7. $(3 - \sqrt{3})^4$
8. $\left(t + \frac{1}{t}\right)^7$
9. $(\sqrt{2} + \sqrt{3})^6$

<u>B</u>. Write, and simplify, the indicated term.

<u>Sample</u>. 7th term of the binomial expansion of '$(2x - y)^{12}$'

<u>Solution</u>. $C(12, 6)(2x)^{12-6}(-y)^6$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} 2^6x^6y^6$$

$$= 2^8 \cdot 3 \cdot 7 \cdot 11x^6y^6 = 59136x^6y^6$$

1. 6th term; $(1 - 2x)^{11}$
2. 4th term; $(3x + y)^{10}$
3. 8th term; $(a^2 + bc)^9$
4. 10th term; $(3u - 6v)^{13}$
5. middle term; $\left(\frac{x}{2} + \frac{y}{3}\right)^6$
6. "constant term"; $\left(x + \frac{2}{x}\right)^8$
7. 17th term; $(6 + 10x)^{16}$
8. 2nd from last term; $(p + 2q)^{97}$

9. term involving 'x^5' [that is, the term which, when simplified, contains 'x^5']; $\left(\frac{x^2}{2} - \frac{2}{x}\right)^{10}$

10. term involving 'y^6'; $(1 + 2y^2)^7$

C. Use the binomial theorem to compute each of the following correct to the nearest 0.001.

1. $(1.02)^{27}$ [Hint. $1.02 = 1 + 0.02$]
2. $(0.99)^{12}$
3. $(1.03)^{10}$
4. $(0.98)^{20}$

D. Use the binomial theorem to expand each of the following.

1. $(x + y + z)^2$
2. $(1 + a + b)^3$
3. $(1 + x + x^2)^4$

E. Use the binomial theorem to prove each of the following.

1. Theorem 176. [Hint. Expand '$(1 + 1)^j$'.]
2. Theorem 177.
3. $\forall_{x \geq 0} \forall_{k \geq 0} (1 + x)^k \geq 1 + kx$ [Compare with Theorem 162.]

F. Here is the start of an inductive proof for Theorem 178. Explain the steps, and tell what is needed to complete the proof.

Part (i): $(a + b)^0 = 1 = C(0, 0)a^{0-0}b^0 = \sum_{k=0}^{0} C(0, k)a^{0-k}b^k$

Part (ii): Suppose [for some $j \geq 0$] that

$$(a + b)^j = \sum_{k=0}^{j} C(j, k)a^{j-k}b^k.$$

Since $(a + b)^{j+1} = (a + b)(a + b)^j$, it follows that

$$(a + b)^{j+1} = (a + b) \sum_{k=0}^{j} C(j, k)a^{j-k}b^k$$

$$= a\sum_{k=0}^{j} C(j, k)a^{j-k}b^{k} + b\sum_{k=0}^{j} C(j, k)a^{j-k}b^{k}$$

$$= \sum_{k=0}^{j} C(j, k)a^{(j+1)-k}b^{k} + \sum_{k=0}^{j} C(j, k)a^{j-k}b^{k+1}$$

$$= \sum_{k=0}^{j} C(j, k)a^{(j+1)-k}b^{k} + \sum_{k=1}^{j+1} C(j, k-1)a^{(j+1)-k}b^{k}$$

$$= C(j, 0)a^{(j+1)-0}b^{0} + \sum_{k=1}^{j} C(j, k)a^{(j+1)-k}b^{k}$$

$$+ \sum_{k=1}^{j} C(j, k-1)a^{(j+1)-k}b^{k} + C(j, j)a^{(j+1)-(j+1)}b^{j+1}$$

$$= a^{(j+1)-0}b^{0} + \sum_{k=1}^{j} [C(j, k) + C(j, k-1)]a^{(j+1)-k}b^{k} + a^{(j+1)-(j+1)}b^{j+1}$$

$$= C(j+1, 0)a^{(j+1)-0}b^{0} + \sum_{k=1}^{j} C(j+1, k)a^{(j+1)-k}b^{k}$$

$$+ C(j+1, j+1)a^{(j+1)-(j+1)}b^{j+1}$$

$$= \sum_{k=0}^{j+1} C(j+1, k)a^{(j+1)-k}b^{k}.$$

* * *

☆In some applications of the binomial theorem it is desirable to be able to find, for given values of 'x' and 'n', the term of maximum absolute value in the expansion of '$(1 + x)^n$'. This is rather easy to do. To see how, let's compare the absolute values of the pth and (p - 1)th terms. Since, for $1 < p \leq n + 1$,

$$\frac{|C(n, p-1)x^{p-1}|}{|C(n, p-2)x^{p-2}|} = \frac{n-(p-2)}{(p-2)+1}|x| \qquad \text{[Explain.]},$$

it follows that

$$|C(n, p-1)x^{p-1}| \geq |C(n, p-2)x^{p-2}| \iff \frac{(n+2)-p}{p-1}|x| \geq 1.$$

Solving the second inequation for 'p', we see that the absolute value of the pth term is not less than that of the (p - 1)th term if and only if

$$(*) \qquad p \leq \frac{(n+2)|x|+1}{1+|x|}.$$

It follows that the term of largest absolute value is the p_mth term where p_m is the largest positive integer which is not greater than n + 1 and satisfies (*). [When the value of the right side of (*) is a positive integer not greater than n + 1 there are two successive terms which have the same absolute value. In this case the p_mth term is the second of these.] Consequently, p_m is the smaller of

$$\left[\!\left[\frac{(n+2)|x|+1}{1+|x|}\right]\!\right] \text{ and } n+1.$$

Since, in any case, the former is at most n + 1, $p_m = \left[\!\left[\dfrac{(n+2)|x|+1}{1+|x|}\right]\!\right]$.

Example. Find the term of maximum absolute value in the expansion of '$(\sqrt{2}-\sqrt{3})^{10}$'.

Solution. Since $(\sqrt{2}-\sqrt{3})^{10} = (\sqrt{2})^{10}\left(1-\dfrac{\sqrt{3}}{\sqrt{2}}\right)^{10}$, it is sufficient to find the term of maximum absolute value in the expansion of '$\left(1-\sqrt{\dfrac{3}{2}}\right)^{10}$'. This is the p_mth term, where

$$p_m = \left[\!\left[\frac{12\sqrt{\frac{3}{2}}+1}{1+\sqrt{\frac{3}{2}}}\right]\!\right].$$

Since $\sqrt{\dfrac{3}{2}} = \sqrt{\dfrac{3\cdot 2}{2\cdot 2}} = \dfrac{1}{2}\sqrt{6}$, it follows that

$$\frac{12\sqrt{\frac{3}{2}} + 1}{1 + \sqrt{\frac{3}{2}}} = \frac{2(6\sqrt{6} + 1)}{2 + \sqrt{6}}$$

$$= \frac{2(6\sqrt{6} + 1)(\sqrt{6} - 2)}{(\sqrt{6} + 2)(\sqrt{6} - 2)}$$

$$= \frac{2(36 - 11\sqrt{6} - 2)}{6 - 4}$$

$$= 34 - 11\sqrt{6} \doteq 7.06.$$

Consequently, $p_m = 7$. The 7th term in the expansion of '$(\sqrt{2} - \sqrt{3})^{10}$' is '$C(10, 6)(\sqrt{2})^{10-6}(-\sqrt{3})^6$' and its value is 22680.

* * *

☆G. Find the term of largest absolute value.

1. $\left(1 + \frac{1}{2}\right)^{18}$ 2. $(1 + 0.02)^{97}$ 3. $(3 + 4)^{15}$ 4. $(1 - 0.03)^{256}$

☆SUMS OF POWERS

In proving Theorem 131 you found summation theorems for the sequence of [first powers of] the positive integers, the sequence of squares of positive integers, and the sequence of cubes of positive integers. The binomial theorem gives a way of extending this sequence of theorems. To see how, let's find a summation theorem for the sequence of fourth powers of positive integers. We begin by noting that

$$(p - 1)^5 = \sum_{k=0}^{5} C(5, k)p^{5-k}(-1)^k$$

$$= p^5 + \sum_{k=1}^{5} (-1)^k C(5, k)p^{5-k}.$$

From this it follows that

$$p^5 - (p - 1)^5 = -\sum_{k=1}^{5} (-1)^k C(5, k)p^{5-k}.$$

Since, by Theorem 138,

$$\sum_{p=1}^{n} [p^5 - (p-1)^5] = n^5 - 0^5 = n^5,$$

it follows that

$$n^5 = -\sum_{p=1}^{n}\left[\sum_{k=1}^{5}(-1)^k C(5, k)p^{5-k}\right]$$

$$= -\sum_{p=1}^{n}\left[-5p^4 + 10p^3 - 10p^2 + 5p - 1\right]$$

$$= 5\sum_{p=1}^{n} p^4 - \left[10\sum_{p=1}^{n} p^3 - 10\sum_{p=1}^{n} p^2 + 5\sum_{p=1}^{n} p - \sum_{p=1}^{n} 1\right].$$

Consequently,

$$\sum_{p=1}^{n} p^4 = \frac{1}{5}\left(n^5 + 10\sum_{p=1}^{n} p^3 - 10\sum_{p=1}^{n} p^2 + 5\sum_{p=1}^{n} p - \sum_{p=1}^{n} 1\right);$$

and, using the four parts of Theorem 131,

$$\sum_{p=1}^{n} p^4 = \frac{1}{5}\left(n^5 + 10\frac{n^2(n+1)^2}{4} - 10\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2} - n\right)$$

$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

In the same way [starting out by expanding '$(p-1)^{j+1}$'] one can obtain a recursion formula for finding summation theorems for sums of powers. Doing this, one finds that

$$p^{j+1} - (p-1)^{j+1} = -\sum_{k=1}^{j+1}(-1)^k C(j+1, k)p^{j+1-k},$$

from which it follows, by Theorem 138, that [for $j \geq 0$]

$$\text{(i)}\qquad n^{j+1} = -\sum_{p=1}^{n}\left[\sum_{k=1}^{j+1}(-1)^k C(j+1,\ k)p^{j+1-k}\right]$$

$$\text{(ii)}\qquad = -\sum_{k=1}^{j+1}\left[(-1)^k C(j+1,\ k)\sum_{p=1}^{n} p^{j+1-k}\right]$$

$$= (j+1)\sum_{p=1}^{n} p^j - \sum_{k=2}^{j+1}\left[(-1)^k C(j+1,\ k)\sum_{p=1}^{n} p^{j+1-k}\right].$$

Using Theorem 137, this can be written a bit more simply as:

$$n^{j+1} = (j+1)\sum_{p=1}^{n} p^j - \sum_{k=1}^{j}\left[(-1)^{k+1} C(j+1,\ k+1)\sum_{p=1}^{n} p^{j-k}\right]$$

Solving this for '$\sum_{p=1}^{n} p^j$' one obtains the desired recursion formula:

$$(*)\qquad \sum_{p=1}^{n} p^j = \frac{1}{j+1}\left(n^{j+1} + \sum_{k=1}^{j}\left[(-1)^{k+1} C(j+1,\ k+1)\sum_{p=1}^{n} p^{j-k}\right]\right)$$

☆EXERCISES

1. Use formula (*) to obtain, successively, the four parts of Theorem 131.

2. Use formula (*), Theorem 131, and the summation theorem for fourth powers, to complete:

$$\sum_{p=1}^{n} p^5 = \frac{1}{6}($$

3. Simplify the result obtained in Exercise 2.

4. Explain the step from (i) to (ii) in the proof of (*).

☆SUMMATION BY DIFFERENCE-SEQUENCES

On pages 8-55 through 8-65 you studied a method--the method of difference-sequences--for proving summation theorems. This method was based on Theorems 140 and 139. Looking, now, at the four parts of Theorem 139, should suggest to you that binomial coefficients are somehow involved. For example, Theorem 139<u>a</u> could be rewritten:

$$(*) \qquad \forall_n \sum_{p=1}^{n} C(p-1, 1) = C(n, 2)$$

Parts <u>b</u>, <u>c</u>, and <u>d</u> of Theorem 139 are equivalent to theorems of a form much like that of (*). Rewrite these theorems in such a form. [<u>Hint</u>. Before doing so, you will need to apply Theorem 133.]

The results you should have obtained on transforming the parts of Theorem 139 suggest that it should be possible to prove a generalization of which these are instances. Such a generalization is:

$$(**) \qquad \forall_m \forall_n \sum_{p=1}^{n} C(p-1, m-1) = C(n, m) \qquad \text{[Theorem 179a]}$$

[What theorem is the instance for m = 1 equivalent to?] It is easy to prove (**) by using Theorems 130 and 173.

☆EXERCISES

1. Prove (**).

2. Prove: $\forall_{k \geq 0} \forall_n \sum_{p=1}^{n} C(k+p-1, k) = C(n+k, k+1)$ [Theorem 179b]

3. Which one of the theorems you have proved earlier is related to the theorem of Exercise 2 as Theorem 139 is related to (**)?

* * *

Using Theorems 140 and (**) it is possible to reduce the application of the method of difference-sequences to a formula, and to generalize parts <u>a</u> and <u>c</u> of Theorem 141. In fact, the example which ends on page

8-62 suggests the following theorem.

Theorem 180.

For any sequence a whose mth difference-sequence is a constant,

a. $\forall_n \; a_n = a_1 + \sum_{k=1}^{m} C(n-1, k)(\Delta^k a)_1$

and

b. $\forall_n \; \sum_{p=1}^{n} a_p = na_1 + \sum_{k=1}^{m} C(n, k+1)(\Delta^k a)_1$

Notice that, for m = 1, Theorem 180 reduces to parts a and c of Theorem 141. Part (ii) of an inductive proof of Theorem 180a is easy to give once one notices that if a is a sequence whose (m + 1)th difference-sequence is a constant, then Δa is a sequence whose mth difference-sequence is a constant. One can then use the inductive hypothesis to obtain a formula for Δa, substitute in Theorem 140, and use (**) to simplify the result. Theorem 180b is easily derived from Theorem 180a, with the help, again, of (**).

* * *

4. Prove Theorem 180a.

5. Prove Theorem 180b.

* * *

It can be proved that, for each m, if a is the sequence of mth powers then $\Delta^m a$ is the constant sequence whose value is m!. And it is easily proved that, for any sequences a and b, $\Delta(a + b) = \Delta a + \Delta b$, while, if a is a constant then $\Delta(ab) = a\Delta b$. Use these facts in finding summation theorems for the sequences given in the following exercises.

* * *

6. $a_p = p^5$

7. $a_p = 3p^3 - p^2$

8. $a_p = 5 + 8(p - 1) - 2(p - 1)(p - 2)$

9. a: 2, 3, 7, 9, −4, ..., and $\Delta^4 a$ is a constant.

☆PRIME NUMBERS

As you learned in Unit 4, a <u>prime number</u> is a positive integer which has exactly two factors [with respect to I^+]. Alternatively, a prime number is a positive integer other than 1 which is divisible only by itself and 1. A <u>composite number</u> is a positive integer which is neither 1 nor a prime number.

It is easy to prove that each composite number has a [at least one] prime divisor. For, suppose that n is a composite number which has no prime divisor. Being composite, n has a divisor--say m, which is neither 1 nor n. Since $m \neq 1$, m is either prime or composite. But, since $m \mid n$ and n has no prime divisor, m is not prime. Hence, it follows that m is composite. Also, since $m \mid n$ it follows [by Theorem 126b] that each divisor of m is a divisor of n. So, since n has no prime divisor, neither does m. Since $m \neq n$ and $m \mid n$, it follows [by Theorem 126a] that $m < n$. Hence, given any composite number which has no prime divisor, there is a smaller such number--that is, the set of composite numbers which have no prime divisor has no least member. Consequently, by the least number theorem [Theorem 108], there are no such numbers--each composite number has a prime divisor.

One can actually prove a little more:

> <u>Theorem 181.</u>
>
> Each composite number n has a prime divisor p such that $p^2 \leq n$.

For, suppose that n is a composite number. Then, as previously shown, n has a prime divisor. Let p be the <u>least</u> prime divisor of n. Since n is not prime, $p \neq n$ and, since $p \mid n$, it follows that $p < n$. Also, since $p \mid n$, $\frac{n}{p} \in I^+$ and, since $p < n$, $n/p > 1$. Hence, n/p is either prime or composite and, in either case, has a prime divisor--say, q. Since $q \mid (n/p)$, it follows that $q \leq n/p$ and, since $(n/p) \mid n$, it follows that q is a prime divisor of n. Now, suppose that $p^2 > n$. It follows that $n/p < p$ and, since $q \leq n/p$, $q < p$. But, since q is a prime divisor of n, and p is the least prime divisor of n, this is not the case. Hence, $p^2 \not> n$--

that is, $p^2 \leq n$.

Theorem 181 tells you that, when looking for prime divisors of a positive integer, you need test only those prime numbers which are not greater than the square root of the given integer. For example, when looking for the prime divisor of a number less than or equal to 10000, you need only test the prime numbers which are not greater than 100.

Theorem 181 also justifies a systematic method--called *the sieve of Eratosthenes*--for sorting out the prime numbers. Suppose, for example, that you wish to find all prime numbers not greater than 100. One way to do this is to list all the positive integers not greater than 100, and then cross off the entries for 1 and the composite numbers. To do this crossing off systematically, one begins by crossing off the '1'. The next entry lists the least prime number, 2, and one proceeds to cross off each 2nd entry *after* this one. In doing so, one has crossed off all entries for numbers other than 2 which have 2 as a prime divisor. After doing this, the first entry after that for 2 and which is not crossed off lists the least prime number greater than 2. This number is 3, and the next step is to cross off each 3rd entry after this one. [Some, for example, '6', will have been crossed off previously.] The first entry after that for 3 and which is not yet crossed off lists the least prime number greater than 3. This number turns out to be 5, and the next step is to cross off each 5th entry after this one. Continuing in this way, by the time one has found a prime number greater than $\sqrt{100}$ he will have crossed off the entries for 1 and for all composite numbers not greater than 100. The entries which are left will, then, list just the prime numbers which are not greater than 100.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Place the top edge of a blank sheet of paper under the last row of the table and, on it, list the next hundred positive integers. "Sift out" the prime numbers less than or equal to 200. How many prime numbers are not greater than 100? How many others are not greater than 200? Do you think that, in listing the prime numbers, one would sometime reach an end--that is, sometime run out of prime numbers? Is there a greatest prime number? Is the number of primes finite?

The last two questions are, of course, equivalent, and as Euclid showed, the answer to both is 'no'. To see that there are infinitely many primes, it is sufficient to show that, given any finite set of prime numbers, there is a prime number which does not belong to the set. To do this, suppose that, for each $m \leq n$, p_m is a prime number, and consider the positive integer $\prod_{m=1}^{n} p_m + 1$. Since each of the numbers p_m divides $\prod_{m=1}^{n} p_m$ and none of them divides 1, it follows [by Theorem 126e] that none of the given prime numbers divides $\prod_{m=1}^{n} p_m + 1$. But this number is either prime or composite and, so, has a prime divisor. Hence, there is a prime number different from each of the numbers p_m.

Euclid's proof suggests a recursive definition for a sequence p of prime numbers:

$$\begin{cases} p_1 = 2 \\ \forall_n \ p_{n+1} = \text{the least prime divisor of } \prod_{m=1}^{n} p_m + 1 \end{cases}$$

[Compute the first 5 terms of this sequence.] Euclid's proof shows that no prime number occurs twice in this sequence [Explain.], but it seems to be difficult to determine whether each prime number occurs once.

Another proof, related to Euclid's, of the fact that there are infinitely many prime numbers, proceeds by showing that, for each n, there is a prime number greater than n. Discover this proof by considering, for a given n, the prime divisors of $n! + 1$. [Hint. Does $n! + 1$ have a prime divisor less than or equal to n?]

In sifting out the prime numbers not greater than 200 you probably noticed that the prime numbers seem to be "scarcer" the further out you go in the sequence of positive integers. As evidence for this, it is easy to show that, for each n, there are n consecutive composite numbers. In fact, given n, and $m \leq n + 1$, $(n + 1)! + m$ is divisible by m. So, for $1 < m \leq n + 1$, $(n+1)! + m$ is composite.

On the other hand, it can be proved that, for each $n > 1$, there is at least one prime p such that $n < p < 2n$.

Although there is only one pair of consecutive primes [Why?], it seems possible that there are infinitely many pairs of primes which, like 3 and 5, 5 and 7, 11 and 13, etc., differ by 2. Although many people have tried to discover whether this is the case, no one yet knows the answer.

Another of the many unsolved questions concerning primes is the following: Is every even number other than 2 either a sum of two primes or double a prime? That this is the case was conjectured by Goldbach in 1742; but, despite much effort, no one yet knows whether Goldbach's conjecture is correct.

The preceding remarks on the way the prime numbers are distributed among the positive integers suggest that it would be very difficult to find a formula for the prime numbers. In fact, no one has succeeded in doing so, and it seems very unlikely that anyone ever will. Even the simpler task of finding a formula all of whose values are prime numbers is a difficult one [unless one is content with formulas such as '$3n^0$']. For example, although, as is easily checked, $n^2 - n + 41$ is a prime for $n = 1, 2, 3, 4, 5$, and many other positive integers, it is not difficult to find an n such that $n^2 - n + 41$ is composite. [Find the least such positive integer.] In fact, it is not difficult to prove that each such expression as '$n^2 - n + 41$' has a composite value for each of infinitely many values of 'n'. [For a proof, see <u>Number Theory and its History</u> by O. Ore (McGraw-Hill), pages 80 and 81.] Perhaps the closest anyone has yet come to finding a formula each of whose values is prime is the recently proved theorem:

$$\exists_x \forall_n \llbracket x^{3^n} \rrbracket \text{ is a prime}$$

[This was proved in 1947 by W. H. Mills.] What is lacking here is the

knowledge of a particular number c such that, for each n, $[\![c^{3^n}]\!]$ is a prime number.

Returning now to Euclid's theorem that there are infinitely many prime numbers, notice that this means that infinitely many terms of the AP for which $a_1 = 1$ and $d = 1$ are prime numbers. A nineteenth-century mathematician, Dirichlet, generalized this by proving that an AP whose terms are positive integers has infinitely many prime terms if and only if $\text{HCF}(a_1, d) = 1$. The only if-part of this theorem is very easy to prove [Do so.], but, although several proofs have been given for the if-part, no one has yet found a simple one. Still, it is not difficult to prove that the AP for which $a_1 = 3$ and $d = 4$ has infinitely many prime terms. To do so, we begin by recalling that [for this AP], for each n, $a_n = 3 + (n - 1)4 = 4n - 1$. So, the problem is to show that $\{m: \exists_n\, m = 4n - 1\}$ contains infinitely many prime numbers. [Instead of this last, it is customary to say "There are infinitely many primes <u>of the form 4n - 1</u>."] We can do this by using a variation of Euclid's proof. But, before doing so, we need to prove two preliminary results [or: lemmas]:

<u>Lemma 1.</u>

The product of numbers each of which is of the form $4n + 1$ is, itself of the form $4n + 1$.

<u>Lemma 2.</u>

Each composite number is a product of prime numbers.

Lemma 1 is easily proved by induction. To begin with, we note that, since $(4m + 1)(4n + 1) = 4(4mn + m + n) + 1$, the product of two numbers of the form $4n + 1$ [and, also, the square of one such number] is, itself, of the form $4n + 1$. Suppose now, that, for each p, n_p is of the form $4n + 1$. Since $\prod_{q=1}^{1} n_q = n_1$, it follows that $\prod_{q=1}^{1} n_q$ is of the form $4n + 1$. Next, suppose that $\prod_{q=1}^{m} n_q$ is of the form $4n + 1$. Since $\prod_{q=1}^{m+1} n_q = \prod_{q=1}^{m} n_q \cdot n_{m+1}$, and since, by hypothesis, n_{m+1} is of the form $4n + 1$,

it follows, as was shown above, that $\prod_{q=1}^{m+1} n_q$ is of the form $4n + 1$. Hence, by the PMI, if, for each p, n_p is of the form $4n + 1$ then, for each m, $\prod_{q=1}^{m} n_q$ is of the form $4n + 1$.

To prove Lemma 2, suppose that n is a composite number which is not a product of prime numbers. Since n is composite it has a divisor--say, m--which is neither 1 nor n. Since $m \mid n$, it follows that $\frac{n}{m}$ is a positive integer which is also a divisor of n and is neither 1 nor n. Since both m and $\frac{n}{m}$ are divisors of n, and neither is n, it follows that both are less than n. Since neither is 1, each is either prime or composite. If each of m and $\frac{n}{m}$ is a prime number or a product of prime numbers, then so is n. Hence, since, by hypothesis, n is not a product of primes, at least one of the numbers m and $\frac{n}{m}$ is neither a prime nor a product of primes. So, at least one of them is a composite number less than n which is not a product of primes. Hence, given any composite number which is not a product of primes, there is a smaller such number. Consequently, by the least number theorem, there is no such number--each composite number is a product of prime numbers.

We are now ready to prove that, given any finite set of prime numbers each of the form $4n - 1$, there is a prime number of the form $4n - 1$ which does not belong to this set. To do so, suppose that, for each $q \leq m$, p_q is a prime number of the form $4n - 1$, and consider the positive integer $4 \cdot \prod_{q=1}^{m} p_q - 1$. This number is certainly greater than 1 and, so, is either prime or composite. If it is prime, it is a prime number of the form $4n - 1$ which is not any of the numbers p_q [Explain.]. On the other hand, if it is composite then, by Lemma 2, it is a product of primes. Since it is an odd number, it is not divisible by 2 and, so, is a product of odd primes. Now, each odd number [and, in particular, each odd prime] is either of the form $4n + 1$ or of the form $4n - 1$ [Explain.].

If $4 \cdot \prod_{q=1}^{m} p_q - 1$ were a product of primes of the form 4n + 1 then, by Lemma 1, it would, itself be of this form. Since it is not, it follows that $4 \cdot \prod_{q=1}^{m} p_q - 1$ has a prime divisor of the form 4n - 1. Since none of the numbers p_q is a divisor of $4 \cdot \prod_{q=1}^{m} p_q - 1$, it follows that there is a prime of the form 4n - 1 which differs from each of the numbers p_q. Consequently, given any finite set of prime numbers of the form 4n - 1, there is a prime of the form 4n - 1 which does not belong to this set--that is, there are infinitely many prime numbers of the form 4n - 1.

In the same way, one can prove that there are infinitely many prime numbers of the form 6n - 1--that is, that the AP for which $a_1 = 5$ and $d = 6$ has infinitely many prime terms.

We shall complete our discussion of prime numbers by proving a stronger theorem than Lemma 2:

> <u>Theorem 183.</u>
>
> Each positive integer other than 1 has a unique prime factorization.

Theorem 183 means that, given any positive integer $n \neq 1$, there is just one finite sequence, $p_1, p_2, \ldots, p_m$, of prime numbers such that, for each $i < m$, $p_i \leq p_{i+1}$, and $\prod_{i=1}^{m} p_i = n$. Evidently, each <u>prime</u> number has a [unique] prime factorization and, using Lemma 2 [together with the apm and the cpm], it can be shown that each composite number has <u>at least</u> one prime factorization. So, all that remains to be proved is that if $p_1, p_2, \ldots, p_m$ are prime numbers such that, for each $i < m$, $p_i \leq p_{i+1}$ and $q_1, q_2, \ldots, q_{m_1}$ are prime numbers such that, for each $j < m_1$, $q_j \leq q_{j+1}$, and

$$\prod_{i=1}^{m} p_i = \prod_{j=1}^{m_1} q_j,$$

then $m_1 = m$ and, for each $i \leq m$, $q_i = p_i$.

We shall prove this, just as we proved Lemma 2, by showing that, given a positive integer n which has two prime factorizations, there is a positive integer smaller than n which has two prime factorizations. From this it follows that there is no smallest positive integer which has two prime factorizations--whence, by the least number theorem there is no such number at all.

In proving this we shall use a consequence of Theorem 128 [see Theorem 182, below]. Recall that two positive integers which have no common divisor other than 1 are said to be <u>relatively prime</u> [and each is said to be prime to the other]. For example, if p is a prime and is not a divisor of some positive integer q then p and q are relatively prime [For, since the only divisors of p are 1 and p, it follows that if p is not a divisor of q then the only <u>common</u> divisor of p and q is 1.]. Now, Theorem 128 tells us that a positive integer which divides the product of two positive integers and is prime to one of them must be a divisor of the other. So, in particular, a prime number which divides the product of two positive integers and is not a divisor of one of them must be a divisor of the other. In other words;

(*) $\left\{ \begin{array}{l} \text{each prime divisor of a product of two positive integers} \\ \text{is a divisor of [at least] one of these integers.} \end{array} \right.$

Using (*) it is easy to prove:

<u>Theorem 182.</u>

For any sequence n of positive integers, and any prime number p,

$$\forall_m \left[p \mid \prod_{i=1}^{m} n_i \Longrightarrow \exists_{q \leq m}\, p \mid n_q \right]$$

[The case m = 2 is just another statement of (*).]

To prove Theorem 182, suppose that n is a sequence of positive integers and that p is a prime number. In the first place, since $\prod_{i=1}^{1} n_i = n_1$, it follows that if $p \mid \prod_{i=1}^{1} n_i$ then $p \mid n_1$ and, so, there is a $q \leq 1$ such that $p \mid n_q$. Now suppose [inductive hypothesis] that if $p \mid \prod_{i=1}^{m} n_i$ then there is a $q \leq m$ such that $p \mid n_q$, and suppose that $p \mid \prod_{i=1}^{m+1} n_i$. Since $\prod_{i=1}^{m+1} n_i = \prod_{i=1}^{m} n_i \cdot n_{m+1}$, it follows from (*) that, since $p \mid \prod_{i=1}^{m+1} n_i$, either $p \mid \prod_{i=1}^{m} n_i$ or $p \mid n_{m+1}$. In the first case it follows from the inductive hypothesis that there is a $q \leq m$ such that $p \mid n$. So [since if $q \leq m$ then $q \leq m + 1$] it follows that, in either case, there is a $q \leq m + 1$ such that $p \mid n_q$. So, from the inductive hypothesis, it follows that if $p \mid \prod_{i=1}^{m+1} n_i$ then there is a $q \leq m + 1$ such that $p \mid n_q$.

Consequently, Theorem 182 follows by mathematical induction.

We are now ready to prove Theorem 183. To do so, suppose that n is a positive integer which has two prime factorizations--that is, suppose that

$$n = \prod_{i=1}^{m} p_i \quad \text{and} \quad n = \prod_{j=1}^{m_1} q_j;$$

that each p_i and each q_j is a prime number; that, for each $i < m$, $p_i \leq p_{i+1}$; that, for each $j < m_1$, $q_j \leq q_{j+1}$; but that it is not the case that $m_1 = m$ and, for each $i \leq m$, $q_i = p_i$--that is, either $m_1 \neq m$ or $m_1 = m$ and, for some $i \leq m$, $q_i \neq p_i$.

In the first place, notice that it follows from our hypothesis that neither m nor m_1 is 1. For if, say, $m = 1$ and $m_1 \neq 1$ then the prime p_1 would be a product of two or more primes, which is impossible.

And, if $m = 1$ and $m_1 = 1$, then it would follow that $m = m_1$ and, for each $i \leq m$, $p_i = q_i$, which contradicts the hypothesis. Consequently, we see that both $m > 1$ and $m_1 > 1$.

Since $n = \prod_{i=1}^{m} p_i = p_1 \cdot \prod_{i=2}^{m} p_i$, it is clear that $p_1 | n$. Hence, since $n = \prod_{j=1}^{m_1} q_j$, it follows that $p_1 | \prod_{j=1}^{m_1} q_j$. Consequently, by Theorem 182, there is a $j \leq m_1$ such that $p_1 | q_j$. Since p_1 and each of the q_j s are prime numbers, it follows that there is a $j \leq m_1$ such that $p_1 = q_j$. Now, it follows from our hypothesis that for each $j \leq m_1$, $q_1 \leq q_j$. Consequently, $q_1 \leq p_1$. In just the same way [interchanging the ps and the qs] it follows that $p_1 \leq q_1$. Hence, $p_1 = q_1$, and $n/p_1 = n/q_1$. Since $p_1 | n$ and $p_1 > 1$, it follows that n/p_1 is a positive integer smaller than n.

Since $n = \prod_{i=1}^{m} p_i$, it follows [by Theorem 147] that $n = p_1 \cdot \prod_{i=2}^{m} p_i$, and that $n/p_1 = \prod_{i=2}^{m} p_i$. Hence, [by Theorem 148], $n/p_1 = \prod_{i=1}^{m-1} p_{i+1}$. And, since, as shown previously, $m > 1$, $m - 1$ is a positive integer. Similarly, $n/q_1 = \prod_{j=1}^{m_1 - 1} q_{j+1}$, and $m_1 - 1$ is a positive integer.

Since $n/p_1 = n/q_1$, and is a positive integer which is less than n, it follows that there is a positive integer n_1 which is less than n and such that

$$n_1 = \prod_{i=1}^{m-1} p_{i+1} \quad \text{and} \quad n_1 = \prod_{j=1}^{m_1 - 1} q_{j+1}.$$

Moreover, $m - 1$ and $m_1 - 1$ are positive integers and each of the numbers p_{i+1} and q_{j+1} is a prime number. Also, for each $i < m - 1$, $p_{i+1} \leq p_{(i+1)+1}$ and, for each $j < m_1 - 1$, $q_{j+1} \leq q_{(j+1)+1}$. Finally,

recalling that, by hypothesis, either $m_1 \neq m$ or $m_1 = m$ and, for some $i \leq m$, $q_i \neq p_i$, and that, as proved above, $q_1 = p_1$, it follows that either $m_1 - 1 \neq m - 1$ or $m_1 - 1 = m - 1$ and, for some $i \leq m - 1$, $q_{i+1} \neq p_{i+1}$.

Consequently, from the hypothesis that there is a positive integer n which has two prime factorizations, it has followed that there is a positive integer $n_1 < n$ which has two prime factorizations. So, the set of those positive integers which have two prime factorizations has no least member. Hence, by the least number theorem there are no such positive integers.

* * *

You can learn more about prime numbers and other aspects of number theory in the books listed below.

H. Davenport. The Higher Arithmetic: An Introduction to the Theory of Numbers. Harper Torchbook Series. [New York: Harper and Brothers, 1960] 172p.

I. A. Barnett. Some Ideas about Number Theory. [Washington, D. C.: National Council of Teachers of Mathematics, 1961] 71p.

A. O. Gelfond. The Solution of Equations in Integers. Golden Gate Books. [San Francisco: W. H. Freeman, 1961] 63p.

Oystein Ore. Number Theory and its History. [New York: McGraw-Hill, 1948] 370p.

REVIEW EXERCISES

1. Given that, for each p, $a_p = 5p + 3$ and $b_p = 3p - 1$, find

   (a) which corresponding terms of a and b, if any, are equal, and

   ☆(b) which noncorresponding terms are equal.

2. Consider the sequence a such that, for each n, $a_n = (n - 1)(n + 1)$. Find three consecutive terms of a whose sum is 242.

3. Solve.

   (a) $\sum_{p=3}^{n} (2p - 5) = (6 - 5) + (8 - 5)$ (b) $\sum_{p=7}^{n} (9p^2 + 2p + 4) = 0$

   (c) $\sum_{p=n}^{11} \frac{1}{p} = \frac{21}{110} + \sum_{p=n}^{9} \frac{1}{p}$ (d) $\sum_{p=1}^{n+4} 3p = 315$

   (e) $\sum_{p=1}^{n} p < 153$ (f) $\sum_{p=1}^{40} p < n \leq \sum_{p=1}^{41} p$ (g) $\sum_{p=1}^{n} p < 41 \leq \sum_{p=1}^{n+1} p$

4. (a) Prove: $\forall_n \sum_{p=1}^{n} p^2(p + 1) = \frac{n(n + 1)(n + 2)(3n + 1)}{12}$

   (1) by using Theorem 130 (2) by mathematical induction

   (b) Use the result of part (a) and Theorem 132c in proving:

   $$\forall_n \sum_{p=1}^{n} p(p + 1)^2 = \frac{n(n + 1)(n + 2)(3n + 5)}{12}$$

   (c) Use the result of part (a) in proving:

   $$\forall_n \sum_{p=1}^{n} (p - 1)^2 p = \frac{n(n^2 - 1)(3n - 2)}{12}$$

5. Prove:

(a) $0 \cdot 2 + 1 \cdot 3 + 2 \cdot 4 + \ldots + 49 \cdot 51 = 42875$

(b) $0 \cdot 6 + 1 \cdot 7 + 2 \cdot 8 + \ldots + 59 \cdot 65 = 80830$

(c) $5 \cdot 6 + 5 \cdot 8 + 5 \cdot 10 + \ldots + 5 \cdot 1000 = 1252470$

☆6. Find the area-measure of the region bounded by the graphs of '$y = 0$', '$y = x^2 + x$', and '$x = 0$'. [Hint. Use the results of Exercise 7 of Part B on page 8-27.]

7. Solve: $\sum_{p=1}^{n} (p^2 - 1) = 19n$

8. Find the arithmetic mean, $\overline{a}$, of the sequence a such that, for each p, $a_p = p(p + 1)$.

☆9. John had a small bag of marbles. The bag split as he was running home and the marbles scattered in the grass. As he searched for them, John tried to remember how many marbles had been in the bag. He couldn't remember this, but he did remember that when he counted them by twos, there was one left over, when he counted them by threes, there were two left over, and when he counted them by fives, he came out even. How many marbles were in the bag?

10. Use Theorem 138 and the algebra theorem:

$$\forall_p \frac{3}{p(p+1)(p+2)(p+3)} = \frac{1}{p(p+1)(p+2)} - \frac{1}{(p+1)(p+2)(p+3)}$$

to find the sum of the first 100 terms of the sequence

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4}, \quad \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}, \quad \frac{1}{3 \cdot 4 \cdot 5 \cdot 6}, \quad \ldots .$$

11. Find a_{100} given that $a_1 = 12$, $(\Delta a)_2 = 50$, $(\Delta^2 a)_3 = 34$, and $\forall_p (\Delta^3 a)_p = 6$.

12. (a) Given an arithmetic progression a with first term a_1 and common difference d. Derive the formulas for a_n and s_n.

(b) If b is an arithmetic progression such that $b_1 = 12$, $b_2 = 16$, and $b_3 = 20$, find m such that $s_m = 208$.

(c) Suppose that c is an arithmetic progression such that $c_2 = 7.75$, $c_{31} = 0.5$, and, for some n, $c_n = -6.5$. Find c_1 and n.

13. Find x such that $1 + x$, $3 + x$, and $2 + x$ are three consecutive terms of a geometric progression.

14. Find the arithmetic mean and the geometric mean of $7\sqrt{3} + 1$ and $7\sqrt{3} - 1$.

15. Find common fraction equivalents for each of the following repeating decimals.

(a) $0.\overline{91}$ (b) $0.6\overline{45}$ (c) $3.2\overline{573}$ (d) $0.07\overline{3}$

16. Find the sums of these infinite geometric progressions.

(a) $3^{-1}, 3^{-2}, 3^{-3}, \ldots$ (b) $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \ldots$

17. Transform to simple expressions with nonnegative exponents.

(a) $-\dfrac{m^{-1} + y^{-1}}{m^{-2} - y^{-2}}$ (b) $(a + b)^2(a^{-1} + b^{-1})^{-2}$ (c) $\dfrac{b^{-1} + b^{-2}}{b}$

(d) $\dfrac{7^0}{3^{-1} + 4^{-1}}$ (e) $\dfrac{(j + k)^{-1}}{j^{-1} + k^{-1}}$ (f) $\dfrac{1^{p+2}}{2^{-3} - 3^{-2}}$

(g) $b^{m-p} \cdot b^{p-n} \cdot c^{n-m}$ (h) $\dfrac{(x + y)^0 \cdot (x + y^0)}{1 - x^{-2}}$

(i) $\dfrac{6^2x^{-4}y^3z^{-1}}{6^{-4}x^3y^{-2}z^{-3}}$ (j) $\left(\dfrac{a^2}{b^{-3}}\right)^{-2}\left(\dfrac{a^{-3}}{b^{-2}}\right)^3$

18. Solve.

(a) $3^{2k} = 81$ (b) $2^{-5k} = 4^{-5}$ (c) $2^{3k} = 1$

(d) $9^{j+2} = 720 + 9^j$ (e) $4^k - 4^{k-1} = 48$

☆(f) $3^{3k} - 3^{2k+2} - 3^{k+4} + 3^6 = 0$

☆(g) $4^{k+4} - 2^{k+5} - 2^{k+3} + 1 = 0$

19. [For nonzero integers a, b, and c, $a^{b^c} = a^{(b^c)}$.]

(a) Prove: $\forall_{i \geq 0}\ (2^{2^i} + 1)(2^{2^i} - 1) = 2^{2^{i+1}} - 1$

(b) Prove: $\forall_{k \geq -1}\ \prod_{i=0}^{k} (2^{2^i} + 1) = 2^{2^{k+1}} - 1$

20. For any sequence a of nonzero integers,

$$\begin{cases} \mathop{\mathsf{E}}\limits_{p=1}^{0} a_p = 1 \\[2ex] \forall_{k \geq 0}\ \mathop{\mathsf{E}}\limits_{p=1}^{k+1} a_p = \left(a_{k+1}\right)^{\left(\mathop{\mathsf{E}}\limits_{p=1}^{k} a_p\right)} \end{cases}$$

(a) Complete: $\mathop{\mathsf{E}}\limits_{p=1}^{1} a_p =$, $\mathop{\mathsf{E}}\limits_{p=1}^{2} a_p =$, $\mathop{\mathsf{E}}\limits_{p=1}^{3} a_p =$

(b) Compute $\mathop{\mathsf{E}}\limits_{p=1}^{4} a_p$ for the sequence a such that $\forall_p\ a_p = 2$.

(c) Prove: $a_1 = 1 \Rightarrow \forall_{k \geq 0}\ \mathop{\mathsf{E}}\limits_{p=1}^{k+1} a_p = \mathop{\mathsf{E}}\limits_{p=1}^{k} a_{p+1}$

21. Guess and prove a theorem which begins:

$$\forall_n \prod_{p=1}^{n} \left(1 - \frac{1}{p+1}\right) = \frac{?}{?}$$

22. Prove that, for any sequence a,

$$\forall_p\, a_p \in I^+ \Longrightarrow \forall_n \sum_{p=1}^{n} a_p \in I^+.$$

☆23. Prove: $\forall_n \forall_x \forall_{y>0}\, [x > y \Longrightarrow x^n > y^n]$

24. Prove: $\forall_k\, (-1)^k = (-1)^{-k}$

25. Definition:

$$\forall_{x>0} \forall_{y>0} \text{ the harmonic mean of } x \text{ and } y \text{ is } \frac{2xy}{x+y}$$

(a) Prove that the geometric mean of any two positive numbers is between their arithmetic mean and their harmonic mean. [Hint. See Exercise 9 on page 8-131.]

(b) Which of the three means of part (a) is the smallest?

(c) Prove that the arithmetic mean of the arithmetic mean and the geometric mean of two positive numbers is the square of the arithmetic mean of the square roots of the two numbers.

(d) Prove that the harmonic mean of the geometric mean and the harmonic mean of two positive numbers is the square of the harmonic mean of the square roots of the two numbers.

☆26. Show that, given any 16 composite numbers each less than 2500, at least two will have a prime factor in common.

27. (a) A lawn 200 feet square is divided very accurately into squares of side-length 1 foot. Show that there will be the same number of blades of grass in at least two squares.

(b) Does it matter that the lawn is divided "very accurately" and "into squares"? What does matter?

☆28. (a) For each n, let P_n be the number of regions into which a plane is divided by n lines no three of which are concurrent and no two of which are parallel.

(1) Compute P_1, P_2, and P_3.

(2) Find a recursive definition for P. [Hint. An (n + 1)th line which intersects n lines in just n points crosses how many of the regions determined by the n lines?]

(3) Guess an explicit definition for P, and derive it from your recursive definition.

(b) Repeat part (a) for the sequence S, where, for each n, S_n is the number of regions into which space is divided by n planes, no four of which are concurrent and each three of which are concurrent.

29. Prove Theorem 145.

30. In how many ways can 2n people be paired off in couples?

31. What is the smallest number of pennies which can be distributed among six pockets in such a way that no two pockets will contain the same number of pennies?

32. Six people eat dinner together in a restaurant. The total bill is $18.00. They decide to split it so that each person pays an amount which is the average of those paid by his neighbors. [One way to accomplish this is for each person to pay $3.00.] Is there more than one way to do this? Why?

33. (a) Find the maximum value of 'xy' if $x + y = 1$.

(b) Find the minimum value of '$\sqrt{x^2 + y^2}$' if $3x + 4y = 25$.

34. If $xy = b$ and $x^{-2} + y^{-2} = a$ then $(x + y)^2 =$ ______.

35. Two rooms are connected by 2 doorways. The first room has 5 doorways and the second room has 4. In how many ways can a person enter the first room, pass on to the second room and, finally, go out of the second room? In how many ways can a person make the trip described above without going through the same doorway twice? In how many ways can he make this trip if he does not start in the second room?

36. In how many ways can 10 children form a ring

(a) if all face inward?

(b) if they face inward and outward, alternately?

(c) if neither (a) nor (b) is the case?

37. In how many ways can 7 people be arranged in a line so that a given person will not be

(a) at either end? (b) in the middle? (c) not in the middle?

38. In how many ways can 3 people be chosen from a group of 10 if

(a) at least one of two given people must be chosen?

(b) both of two given people must be chosen?

(c) neither of two given people may be chosen?

(d) at most one of two given people may be chosen?

39. In how many ways can 10 people be seated in a row if

(a) two given people must be neighbors?

(b) two given people may not be neighbors?

40. The sum of the first 10 terms of a given sequence a is 49. If a new sequence is formed by subtracting 4 from each term of a, multiplying the resulting terms by 5, and then subtracting 4 from each of these terms, what is the sum of the first 10 terms of the new sequence?

41. Expand and simplify: $(\sqrt{3} - \sqrt{2})^6$

42. Which term in the expansion of '$(x^5 - \frac{1}{x^2})^{18}$', when simplified, does not contain 'x'?

43. Find, and simplify, the 10th term in the expansion of '$(x - 2y)^{17}$'.

44. Factor.

(a) $4y^3(x - 3y) - y$

(b) $1 - 4xy - x^2 - 4y^2$

(c) $x^5 + z^{-5}$

(d) $x^2 + x^{-2} - 2$

(e) $n^2 - n + 2\sum_{p=1}^{10} p$

(f) $n^2 + n + 2\sum_{p=1}^{n} p$

(g) $21 + x - 2x^2$

(h) $(x - 2)^2 - x^4$

(i) $x^3 - y^2 + x^2 - y^3$

(j) $a^8 - b^8$

(k) $4a^5 - 128$

(ℓ) $\pi R^2 - \pi r^2$

(m) $x^2y^3 - x^4y$

(n) $x^3 + 2x^2 + x$

☆45. If a '15' is inserted between the digits of '16', the numeral '1156' is obtained. If a '15' is inserted between the middle digits of '1156', the numeral '111556' is obtained. Consider the sequence a whose terms are named in the manner just described.

16, 1156, 111556, 11115556, ...

Prove that each term of this sequence is a perfect square, and discover a formation rule for naming the terms of the sequence $\sqrt{a}$.

☆46. Study the pattern of formation of the first five terms of the sequence a.

$$a_1 = 1$$
$$a_2 = 2 + 6$$
$$a_3 = 3 + 9 + 15$$
$$a_4 = 4 + 12 + 20 + 28$$
$$a_5 = 5 + 15 + 25 + 35 + 45$$

Guess a formula for the nth term and prove your guess correct.

47. Solve: $2^{2k} - 21 \circ 2^k + 80 = 0$

48. Simplify.

(a) $3(\frac{1}{3} + 2)$ (b) $(4 - 2)(3 - 5)$ (c) $(0.2)(0.3)$

49. At what time between 2 and 3 o'clock will the hands be pointing in the same direction?

☆50. A point moves in the number plane, starting at (0, 0), in such a way that, for each n, at the end of the nth second it is at (x_n, y_n), where

$$x_n = x_{n-1} + 3n \quad \text{and} \quad y_n = y_{n-1} + (-1)^{n-1} \cdot 4n.$$

(a) What is the shortest total distance it can have moved during the first n seconds?

(b) Where is the point at the end of the first 2n seconds?

51. Solve these equations.

(a) $x^2 + 4^2 = 5^2$ (b) $3^2 + x^2 = 4^2$ (c) $2x + \frac{1}{2} = 5$

52. Simplify.

(a) $\frac{2 + 3}{2 + 9}$ (b) $\frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13}$ (c) $\frac{1}{2} \cdot \frac{3}{5} + \frac{3}{2} \cdot \frac{6}{5}$

53. Two buckets are of the same shape, but one is 1 foot deep and the other is 8 inches deep. If the larger one holds 10 gallons of water, how much does the smaller one hold?

54. Find the area and the lengths of the diagonals of a parallelogram whose sides are 8 inches and 12 inches long, and one of whose angles is an angle of 156°.

55. The diagonals of a rhombus are 18 and 30 inches long, respectively. Find the measures of the angles of the rhombus.

56. The base, $\overleftrightarrow{AB}$, of isosceles triangle ABC is 20 inches long, and the vertex angle is an angle of 48°. How far from C is the centroid of the triangle?

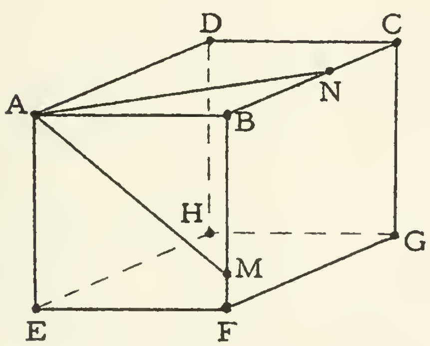
57. Recalling that the trigonometric ratios have been defined only for acute angles, solve each of the following equations.

(a) $\cos (x + 10)^\circ = \sin (x - 30)^\circ$ (b) $\sin (4x)^\circ = \cos x^\circ$

(c) $\tan (3x + 6)^\circ = 1$ (d) $2\sin (5x)^\circ = 1$ (e) $2\cos (3x)^\circ = 1$

58. Find the measure of the angle which contains the diagonal of a cube and the diagonal of a face of the cube.

☆59.



Given: ABCD - EFGH is a cube, $m(\angle NAB) = 30$, $m(\angle MAB) = 40$

Find: $m(\angle NAM)$

60. If $a - b = b - a$ and $\frac{1}{a} = 7$ then $\frac{a}{b} =$.

61. Suppose that A, B, and C are noncollinear points and that $D \in \overrightarrow{BC}$ and $E \in \overrightarrow{AC}$. If $m(\angle ABC) + m(\angle CAB) = 3 \cdot m(\angle ACB)$, what is $m(\angle CDE) + m(\angle CED)$?

BASIC PRINCIPLES AND THEOREMS

Commutative principles for addition and multiplication

$\forall_x \forall_y\ x + y = y + x$ $\qquad\qquad$ $\forall_x \forall_y\ xy = yx$

Associative principles for addition and multiplication

$\forall_x \forall_y \forall_z\ x + y + z = x + (y + z)$ $\qquad\qquad$ $\forall_x \forall_y \forall_z\ xyz = x(yz)$

Distributive principle [for multiplication over addition]

$\forall_x \forall_y \forall_z\ (x + y)z = xz + yz$

Principles for 0 and 1

$\forall_x\ x + 0 = x$ $\qquad$ $\forall_x\ x1 = x$ $\qquad$ $1 \neq 0$

Principle of Opposites

$\forall_x\ x + -x = 0$

Principle for Subtraction

$\forall_x \forall_y\ x - y = x + -y$

Principle of Quotients

$$\forall_x \forall_{y \neq 0}\ \frac{x}{y} y = x$$

* * *

1. $\forall_x \forall_y \forall_z\ x(y + z) = xy + xz$ [page 2-60]

2. $\forall_x\ 1x = x$ [2-61]

3. $\forall_x \forall_a \forall_b \forall_c\ ax + bx + cx = (a + b + c)x$ [2-61]

4. $\forall_x \forall_y \forall_a \forall_b\ (ax)(by) = (ab)(xy)$ [2-61]

5. $\forall_x \forall_y \forall_a \forall_b\ (a + x) + (b + y) = (a + b) + (x + y)$ [2-61]

6. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow x + z = y + z]$ [2-64]

7. $\forall_x \forall_y \forall_z\ [x + z = y + z \Longrightarrow x = y]$ [2-65]

8. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow z + x = z + y]$ [2-66]

9. $\forall_x \forall_y \forall_z\ [z + x = z + y \Longrightarrow x = y]$ [page 2-66]

10. $\forall_x \forall_y\ [x = y \Rightarrow -x = -y]$ [2-66]

11. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow xz = yz]$ [2-66]

12. $\forall_x \forall_y \forall_z\ [x = y \Longrightarrow zx = zy]$ [2-66]

13. $\forall_u \forall_v \forall_x \forall_y\ [(u = v \text{ and } x = y) \Longrightarrow u + x = v + y]$ [2-66]

14. $\forall_u \forall_v \forall_x \forall_y\ [(u = v \text{ and } u + x = v + y) \Longrightarrow x = y]$ [2-66]

15. $\forall_x\ x0 = 0$ [2-66]

16. $\forall_x \forall_y\ [x + y = 0 \Longrightarrow -x = y]$ [2-68]

17. $\forall_x\ --x = x$ [2-69]

18. $\forall_x \forall_y\ -(x + y) = -x + -y$ [2-69]

19. $\forall_x \forall_y\ -(x + -y) = y + -x$ [2-69]

20. $\forall_x \forall_y\ -(xy) = x \cdot -y$ [2-69]

21. $\forall_x \forall_y\ -(xy) = -xy$ [2-69]

22. $\forall_x \forall_y\ [x = -y \Rightarrow -x = y]$ [2-69]

23. $\forall_x \forall_y\ -x \cdot -y = xy$ [2-70]

24. $\forall_x \forall_y\ -xy = x \cdot -y$ [2-70]

25. $\forall_x \forall_y \forall_z\ -x(y + z) = -(xy) + -(xz)$ [2-70]

26. $\forall_x \forall_y \forall_z\ -x(-y + -z) = xy + xz$ [2-70]

27. $\forall_x\ x \cdot -1 = -x$ [2-70]

28. $\forall_x\ -x = -1x$ [2-70]

29. $\forall_x \forall_y\ (x + y) + -y = x$ [2-71]

30. $\forall_x \forall_y\ (x + y) - y = x$ [2-71]

31. $\forall_x \forall_y \forall_z\ x - yz = x + -yz$ [page 2-71]

32. $\forall_x \forall_y\ x - y + y = x$ [2-72]

33. $\forall_x \forall_y\ -(x - y) = y - x$ [2-72]

34. $\forall_x \forall_y \forall_z\ x + (y - z) = x + y - z$ [2-73]

35. $\forall_x \forall_y \forall_z\ x - (y + z) = x - y - z$ [2-73]

36. $\forall_x \forall_y \forall_z\ x - (y - z) = x - y + z$ [2-73]

37. $\forall_x \forall_y \forall_z\ x + (y - z) = x - z + y$ [2-73]

38. $\forall_x \forall_y \forall_z\ x(y - z) = xy - xz$ [2-74]

39. $\forall_x \forall_y \forall_z\ (x - y)z = xz - yz$ [2-74]

40. $\forall_x \forall_y \forall_z\ x - (-y - z) = x + y + z$ [2-74]

41. $\forall_x \forall_y \forall_z \forall_u\ x - (y - z - u) = x - y + z + u$ [2-74]

42. $\forall_x\ 0 - x = -x$ [2-75]

43. $\forall_x\ x - 0 = x$ [2-75]

44. $\forall_x \forall_y \forall_z\ x + z - (y + z) = x - y$ [2-75]

45. $\forall_x \forall_y \forall_z\ x - z - (y - z) = x - y$ [2-75]

46. $\forall_a \forall_b \forall_c \forall_d\ (a - b) + (c - d) = (a + c) - (b + d)$ [2-75]

47. $\forall_x \forall_y \forall_z\ [z + y = x \Longrightarrow z = x - y]$ [2-89]

48. $\forall_x \forall_y \forall_{z \neq 0}\ [xz = yz \Longrightarrow x = y]$ [2-90]

49. $\forall_x \forall_{y \neq 0} \forall_z\ [zy = x \Longrightarrow z = \frac{x}{y}]$ [2-91]

50. $\forall_x\ \frac{x}{1} = x$ [2-91]

51. $\forall_{x \neq 0}\ \frac{x}{x} = 1$ [2-91]

52. $\forall_x\ \frac{x}{-1} = -x$ [2-91]

53. $\forall_{x \neq 0}\ \frac{0}{x} = 0$ [page 2-91]

54. $\forall_x \forall_{y \neq 0}\ [\frac{x}{y} = 0 \Longrightarrow x = 0]$ [2-91]

55. $\forall_x \forall_y\ [(x \neq 0 \text{ and } y \neq 0) \Longrightarrow xy \neq 0]$ [2-91]

56. $\forall_x \forall_y\ [xy = 0 \Longrightarrow (x = 0 \text{ or } y = 0)]$ [2-91]

57. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}$ [2-92]

58. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} - \frac{u}{v} = \frac{xv - uy}{yv}$ [2-92]

59. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} \cdot \frac{u}{v} = \frac{xu}{yv}$ [2-93]

60. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y}$ [2-94]

61. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{x}{y} = \frac{x \div z}{y \div z}$ [2-95]

62. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{xy}{z} = \frac{x}{z}y$ [2-96]

63. $\forall_x \forall_{y \neq 0}\ \frac{x}{y} = x \cdot \frac{1}{y}$ [2-97]

64. $\forall_x \forall_{y \neq 0}\ \frac{xy}{y} = x$ [2-97]

65. $\forall_{x \neq 0} \forall_y \forall_z\ \frac{xy + xz}{x} = y + z$ [2-97]

66. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \forall_{z \neq 0}\ \frac{xu}{yv} = \frac{(x \div z)u}{(y \div z)v}$ [2-98]

67. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x + y}{z}$ [2-99]

68. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} + \frac{u}{vz} = \frac{xv + uy}{yvz}$ [2-99]

69. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} - \frac{y}{z} = \frac{x - y}{z}$ [2-100]

70. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} - \frac{u}{vz} = \frac{xv - uy}{yvz}$ [2-100]

71. $\forall_x \forall_y \forall_{z \neq 0}\ x + \frac{y}{z} = \frac{xz + y}{z}$ [2-100]

72. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ x \div \frac{y}{z} = x\frac{z}{y}$ [page 2-101]

73. $\forall_x \forall_{y \neq 0} \forall_{u \neq 0} \forall_{v \neq 0}\ \frac{x}{y} \div \frac{u}{v} = \frac{xv}{yu}$ [2-101]

74. $\forall_{x \neq 0} \forall_{y \neq 0}\ \frac{1}{x/y} = \frac{y}{x}$ [2-101]

75. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{x}{y} \div z = \frac{x}{yz}$ [2-101]

76. $\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{-x}{y}$ [2-103]

77. $\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{x}{-y}$ [2-103]

78. $\forall_x \forall_{y \neq 0}\ \frac{-x}{-y} = \frac{x}{y}$ [2-103]

79. $\forall_x\ [x \neq 0 \Longrightarrow -x \neq 0]$ [7-18]

80. $-0 = 0$ [7-18]

* * *

(P_1) $\forall_x\ [x \neq 0 \Longrightarrow$ either $x \in P$ or $-x \in P]$ [7-22]

(P_2) $\forall_x$ not both $x \in P$ and $-x \in P$ [7-22]

(P_3) $\forall_x \forall_y\ [(x \in P$ and $y \in P) \Longrightarrow x + y \in P]$ [7-23]

(P_4) $\forall_x \forall_y\ [(x \in P$ and $y \in P) \Longrightarrow xy \in P]$ [7-24]

* * *

81. $0 \notin P$ $[\forall_{x \in P}\ x \neq 0]$ [7-23]

82. $1 \in P$ $[1 > 0]$ [7-23]

* * *

(G) $\forall_x \forall_y\ [y > x \Longleftrightarrow y - x \in P]$ [7-30]

* * *

83. $\forall_x\ [x > 0 \Longleftrightarrow x \in P]$ [7-30]

84. $\forall_x \forall_y\ [y > x \Longleftrightarrow y - x > 0]$ [7-31]

85. $\forall_x\ [x < 0 \iff -x > 0]$ [page 7-32]

86 a. $\forall_x \forall_y\ [x \neq y \implies (x > y \text{ or } y > x)]$

b. $\forall_x \forall_y$ not both $x > y$ and $y > x$

c. $\forall_x \forall_y \forall_z\ [(x > y \text{ and } y > z) \implies x > z]$ [7-32]

d. $\forall_x \forall_y \forall_z\ [x > y \implies x + z > y + z]$

e. $\forall_x \forall_y \forall_z\ [(z > 0 \text{ and } x > y) \implies xz > yz]$

87. $\forall_x\ x \not> x$ $[\forall_x \forall_y\ (x = y \implies x \not> y)]$ [7-33]

88. $\forall_x \forall_y\ [y \geq x \iff x \not> y]$ [7-33]

89. $\forall_x \forall_y \forall_z\ [x + z > y + z \iff x > y]$ [7-33]

90. $\forall_x\ x + 1 > x$ [7-35]

91. $\forall_x \forall_y \forall_u \forall_v\ [(x > y \text{ and } u > v) \implies x + u > y + v]$ [7-35]

92. $\forall_x \forall_y \forall_z\ [(x > y \text{ and } y \geq z) \implies x > z]$ [7-35]

93. $\forall_x \forall_y\ [(x \geq y \text{ and } y \geq x) \implies x = y]$ [7-35]

94. $\forall_x \forall_y\ [-x > -y \iff y > x]$ [7-35]

95 a. $\forall_x \forall_y \forall_{z > 0}\ [xz > yz \iff x > y]$

b. $\forall_x \forall_y \forall_{z < 0}\ [xz < yz \iff x > y]$ [7-36]

96 a. $\forall_x \forall_y\ [xy > 0 \iff ([x > 0 \text{ and } y > 0] \text{ or } [x < 0 \text{ and } y < 0])]$

b. $\forall_x \forall_y\ [xy < 0 \iff ([x > 0 \text{ and } y < 0] \text{ or } [x < 0 \text{ and } y > 0])]$

[7-36]

97 a. $\forall_{x \neq 0}\ x^2 > 0$ [7-38]

b. $\forall_x \forall_y\ [x \neq y \implies x^2 + y^2 > 2xy]$ [7-39]

c. $\forall_{x > 0}\ x + \frac{1}{x} \geq 2$ [7-40]

98 a. $\forall_{x \geq 0} \forall_{y \geq 0} [x^2 = y^2 \Rightarrow x = y]$

b. $\forall_x \forall_{y \geq 0} [y^2 > x^2 \Rightarrow y > x > -y]$

c. $\forall_{x \geq 0} \forall_y [y > x \Rightarrow y^2 > x^2]$ [page 7-38]

99 a. $\forall_x \forall_y \forall_{z \neq 0} [\frac{x}{z} > \frac{y}{z} \Longleftrightarrow xz > yz]$

b. $\forall_{x \neq 0} ([\frac{1}{x} > 0 \Longleftrightarrow x > 0]$ and $[\frac{1}{x} < 0 \Longleftrightarrow x < 0])$ [7-41]

100. $\forall_{x > 0} \forall_{y \neq 0} \forall_{z > 0} [y > z \Rightarrow \frac{x}{z} > \frac{x}{y}]$ [7-41]

* * *

[domain of 'm', 'n', 'p', and 'q' is I^+]

(I_1^+) $1 \in I^+$

(I_2^+) $\forall_n\ n + 1 \in I^+$

(I_3^+) $\forall_S [(1 \in S$ and $\forall_n [n \in S \Rightarrow n + 1 \in S]) \Rightarrow \forall_n\ n \in S]$ [7-49]

* * *

101. $I^+ \subseteq P$ $[\forall_n\ n \in P]$ [7-49]

102. $\forall_m \forall_n\ m + n \in I^+$ [7-56]

103. $\forall_m \forall_n\ mn \in I^+$ [7-56]

104. $\forall_n\ n \geq 1$ [7-84]

105. $\forall_m \forall_n [n > m \Rightarrow n - m \in I^+]$ [7-84]

106. $\forall_m \forall_n [n \geq m + 1 \Longleftrightarrow n > m]$ [7-86]

107 a. $\forall_n\ n \not< 1$

b. $\forall_m \forall_n [n < m + 1 \Longleftrightarrow n \leq m]$ [7-86]

108. Each nonempty set of positive integers has a least member.
$[\forall_S [\emptyset \neq S \subseteq I^+ \Rightarrow \exists_{m \in S} \forall_{n \in S}\ m \leq n]]$ [page 7-88]

* * *

(C) $\forall_x \exists_n\ n > x$ [7-89]

(I) $\forall_x [x \in I \Longleftrightarrow (x \in I^+ \text{ or } x = 0 \text{ or } -x \in I^+)]$ [7-94]

* * *

109. $\forall_x [x \in I \Longleftrightarrow \exists_m \exists_n\ x = m - n]$ [7-94]

*

[domain of 'i', 'j', and 'k' is I]

*

110 a. $\forall_j\ -j \in I$

b. $\forall_j \forall_k\ k + j \in I$

c. $\forall_j \forall_k\ k - j \in I$

d. $\forall_j \forall_k\ kj \in I$

[7-95]

111. $\forall_j \forall_k [k > j \Longleftrightarrow k - j \in I^+]$ [7-96]

112. $\forall_j \forall_k [k + 1 > j \Longleftrightarrow k \geq j]$ [7-96]

113. Each nonempty set of integers which has a lower bound has a least member. [7-98]

114. $\forall_j \forall_S [(j \in S \text{ and } \forall_{k \geq j} [k \in S \Rightarrow k + 1 \in S]) \Rightarrow \forall_{k \geq j}\ k \in S]$ [7-99]

115. Each nonempty set of integers which has an upper bound has a greatest member. [7-100]

116. $\forall_j \forall_S [(j \in S \text{ and } \forall_{k \leq j} [k \in S \Rightarrow k - 1 \in S]) \Rightarrow \forall_{k \leq j}\ k \in S]$ [7-100]

117. $\forall_S [(0 \in S \text{ and } \forall_k [k \in S \Rightarrow (k + 1 \in S \text{ and } k - 1 \in S)]) \Rightarrow \forall_k k \in S]$ [page 7-100]

* * *

$\forall_x \; [\![x]\!] =$ the greatest integer k such that $k \leq x$ [7-102]

$\forall_x \; \{\!\{ x \}\!\} = x - [\![x]\!]$ [7-107]

* * *

118 a. $\forall_x \forall_k [k \leq [\![x]\!] \Longleftrightarrow k \leq x]$ [7-103]

b. $\forall_x \forall_k [k > [\![x]\!] \Longleftrightarrow k > x]$ [7-104]

c. $\forall_x \forall_k [k \geq [\![x]\!] \Longleftrightarrow k + 1 > x]$

d. $\forall_x \forall_k [k < [\![x]\!] \Longleftrightarrow k + 1 \leq x]$

e. $\forall_x \forall_k [k = [\![x]\!] \Longleftrightarrow k \leq x < k + 1]$

(c.–e.) [7-105]

119. $\forall_x \forall_j \; [\![x + j]\!] = [\![x]\!] + j$ [7-105]

120. $\forall_x \forall_{y>0} \exists_k \exists_z \; [x = ky + z \text{ and } 0 \leq z < y]$ [7-106]

121. $\forall_x \forall_{y>0} \exists_n \; ny > x$ [7-106]

122. $\forall_x \; -[\![-x]\!] =$ the least integer k such that $k \geq x$ [7-106]

123. $\forall_x \forall_m \left(\left[\!\!\left[\frac{x}{m} \right]\!\!\right] = \left[\!\!\left[\frac{[\![x]\!]}{m} \right]\!\!\right] \text{ and } \left[\!\!\left[\left\{\!\!\left\{ \frac{x}{m} \right\}\!\!\right\} m \right]\!\!\right] = \left\{\!\!\left\{ \frac{[\![x]\!]}{m} \right\}\!\!\right\} m \right)$ [7-111]

124. $\forall_x \forall_m \; 0 \leq \left\{\!\!\left\{ \frac{[\![x]\!]}{m} \right\}\!\!\right\} m = [\![x]\!] - \left[\!\!\left[\frac{x}{m} \right]\!\!\right] m < m$ [7-111]

* * *

$\forall_m \forall_j \; [m \mid j \Longleftrightarrow \exists_k \; j = mk]$ [7-115 and 7-129]

* * *

125. $\forall_n \; (1 \mid n \text{ and } n \mid n)$ [7-115]

126 a. $\forall_m \forall_n [m | n \Longrightarrow m \le n]$

b. $\forall_m \forall_n \forall_p [(m | n \text{ and } n | p) \Longrightarrow m | p]$

c. $\forall_m \forall_n [(m | n \text{ and } n | m) \Longrightarrow m = n]$

d. $\forall_m \forall_n \forall_p [(m | n \text{ and } m | p) \Longrightarrow m | n + p]$ [page 7-115]

e. $\forall_m \forall_n \forall_p [(m | n \text{ and } m | n + p) \Longrightarrow m | p]$

f. $\forall_m \forall_n \forall_p [m | n \Longrightarrow mp | np]$ [7-116]

127. $\forall_m \forall_n \exists_i \exists_j \; \mathrm{HCF}(m, n) = mi + nj$ [7-122]

128. $\forall_m \forall_n \forall_k [(\mathrm{HCF}(m, n) = 1 \text{ and } m | nk) \Longrightarrow m | k]$ [7-129]

129. $\forall_m \forall_n [\mathrm{HCF}(m, n) = 1 \Longrightarrow$

$\forall_i \forall_j [mi + nj = 0 \Longleftrightarrow \exists_k (i = nk \text{ and } j = -mk)]]$ [7-129]

* * *

For each $j \in I$ and for each function a whose domain includes $\{k: k \ge j\}$,

$$\begin{cases} \displaystyle\sum_{i=j}^{j-1} a_i = 0 \\ \forall_{k \ge j-1} \displaystyle\sum_{i=j}^{k+1} a_i = \sum_{i=j}^{k} a_i + a_{k+1} \end{cases}$$

[page 8-36]
[An earlier form is on page 8-9]

* * *

130. For any sequences a and b,

$$\left(b_1 = a_1 \text{ and } \forall_n \; b_{n+1} = b_n + a_{n+1}\right) \Longrightarrow \forall_n \sum_{p=1}^{n} a_p = b_n$$ [8-17]

131 a. $\forall_n \sum_{p=1}^{n} 1 = n$ b. $\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}$

c. $\forall_n \sum_{p=1}^{n} p^2 = \frac{n(n+1)(2n+1)}{6}$ d. $\forall_n \sum_{p=1}^{n} p^3 = \frac{n^2(n+1)^2}{4}$ [page 8-24]

132 a. $\forall_n \sum_{p=1}^{n} 1 = n$ b. $\forall_n \sum_{p=1}^{n} p = \frac{n(n+1)}{2}$

c. $\forall_n \sum_{p=1}^{n} p(p+1) = \frac{n(n+1)(n+2)}{3}$ [8-24]

d. $\forall_n \sum_{p=1}^{n} p(p+1)(p+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

133. $\forall_x \forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} xa_i = x \sum_{i=j}^{k} a_i$ [8-39]

134. $\forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} (a_i + b_i) = \sum_{i=j}^{k} a_i + \sum_{i=j}^{k} b_i$ [8-42]

135. $\forall_j \forall_{j_1 \geq j-1} \forall_{k \geq j_1} \sum_{i=j}^{k} a_i = \sum_{i=j}^{j_1} a_i + \sum_{i=j_1+1}^{k} a_i$ [8-44]

136. $\forall_j \forall_{k \geq j} \sum_{i=j}^{k} a_i = a_j + \sum_{i=j+1}^{k} a_i$ [8-44]

137. $\forall_j \forall_{j_1} \forall_{k \geq j-1} \sum_{i=j}^{k} a_i = \sum_{i=j+j_1}^{k+j_1} a_{i-j_1}$ [page 8-44]

138. $\forall_n \sum_{p=1}^{n} (a_{p+1} - a_p) = a_{n+1} - a_1$ [8-53]

139 a. $\forall_n \sum_{p=1}^{n} (p-1) = \frac{n(n-1)}{2}$ [8-55]

b. $\forall_n \sum_{p=1}^{n} (p-1)(p-2) = \frac{n(n-1)(n-2)}{3}$

c. $\forall_n \sum_{p=1}^{n} (p-1)(p-2)(p-3) = \frac{n(n-1)(n-2)(n-3)}{4}$ [8-56]

d. $\forall_n \sum_{p=1}^{n} (p-1)(p-2)(p-3)(p-4) = \frac{n(n-1)(n-2)(n-3)(n-4)}{5}$

* * *

$$\forall_p \ (\Delta a)_p = a_{p+1} - a_p$$ [8-57]

* * *

140. $\forall_n \ a_n = a_1 + \sum_{p=1}^{n-1} (\Delta a)_p$ [8-60]

* * *

A sequence a is an arithmetic progression if and only if the sequence Δa is a constant. The value of Δa is called the common difference of the AP. [8-66]

* * *

141 If a is an AP with common difference d, and, for each n, s_n is the sum of its first n terms, then [page 8-67 and 8-68]

a. $\forall_n\, a_n = a_1 + (n-1)d$, b. $\forall_n \forall_{m \neq n}\; d = \dfrac{a_m - a_n}{m - n}$,

c. $\forall_n\, s_n = \frac{n}{2}[2a_1 + (n-1)d]$, d. $\forall_n\, s_n = \frac{n}{2}(a_1 + a_n)$.

142. $\forall_j \forall_{k \geq j-1} \sum_{i=j}^{k} a_i = \sum_{i=j}^{k} a_{k+j-i}$ [8-72]

143. $\forall_n \left[\forall_{m \leq n}\, a_m < b_m \Longrightarrow \sum_{p=1}^{n} a_p < \sum_{p=1}^{n} b_p \right]$ [8-76]

144. $\forall_x \forall_n \left[\sum_{p=1}^{n} a_p \geq x \Longrightarrow \exists_{m \leq n}\, a_m \geq \frac{x}{n} \right]$ [8-81]

* * *

For each $j \in I$ and for each function a whose domain includes $\{k\colon k \geq j\}$, [8-94]

$$\begin{cases} \prod_{i=j}^{j-1} a_i = 1 \\ \forall_{k \geq j-1} \prod_{i=j}^{k+1} a_i = \prod_{i=j}^{k} a_i \cdot a_{k+1} \end{cases}$$

* * *

$$\forall_{k \geq 0}\; k! = \prod_{p=1}^{k} p$$ [8-98]

* * *

145. $\forall_j \forall_{k \geq j-1} \prod_{i=j}^{k} (a_i b_i) = \prod_{i=j}^{k} a_i \cdot \prod_{i=j}^{k} b_i$ [page 8-99]

146. $\forall_j \forall_{j_1 \geq j-1} \forall_{k \geq j_1} \prod_{i=j}^{k} a_i = \prod_{i=j}^{j_1} a_i \cdot \prod_{i=j_1+1}^{k} a_i$ [8-99]

147. $\forall_j \forall_{k \geq j} \prod_{i=j}^{k} a_i = a_j \cdot \prod_{i=j+1}^{k} a_i$ [8-99]

148. $\forall_j \forall_{j_1} \forall_{k \geq j-1} \prod_{i=j}^{k} a_i = \prod_{i=j+j_1}^{k+j_1} a_{i-j_1}$ [8-99]

149. $\forall_j \forall_{k \geq j-1} \prod_{i=j}^{k} a_i = \prod_{i=j}^{k} a_{k+j-i}$ [8-99]

* * *

$$\begin{cases} \forall_x \forall_{k \geq 0}\ x^k = \prod_{p=1}^{k} x & [8\text{-}100] \\ \forall_{x \neq 0} \forall_{k < 0}\ x^k = \dfrac{1}{x^{-k}} & [8\text{-}114] \end{cases}$$

* * *

150 a. $\forall_k\ 1^k = 1$ [8-103 and 8-115]

b. $\forall_k\ (-1)^{k+2} = (-1)^k$ [8-102]

c. $\forall_k\ [(-1)^{2k} = 1 \text{ and } (-1)^{2k+1} = -1]$ [8-103]

d. $0^0 = 1$ and $\forall_n\ 0^n = 0$ [8-103]

151 a. $\forall_m \forall_{k \geq 0}\ m^k \in I^+$ [page 8-103]

b. $\forall_{m > 1} \forall_{k \geq 0}\ m^k > k$ [8-107]

152 a. $\forall_{x > 0} \forall_k\ x^k > 0$ b. $\forall_{x \neq 0} \forall_k\ x^k \neq 0$

c. $\forall_{x > 1} \forall_k\ x^{k+1} > x^k$

[8-103 and 8-115]

153. $\forall_x \forall_{k \geq 0}\ (x - 1) \sum_{p=1}^{k} x^{p-1} = x^k - 1$ [8-103]

154. $\forall_{x \neq 0} \forall_k\ x^{-k} = \frac{1}{x^k}$ [8-114]

155. $\forall_{x \neq 0} \forall_j \forall_k\ x^j x^k = x^{j+k}$ $[\forall_x \forall_{j \geq 0} \forall_{k \geq 0}\ x^j x^k = x^{j+k}]$ [8-117]

156. $\forall_{x \neq 0} \forall_j \forall_k\ \frac{x^j}{x^k} = x^{j-k}$ [8-118]

157. $\forall_{x \neq 0} \forall_j \forall_k\ (x^j)^k = x^{jk}$ $[\forall_x \forall_{j \geq 0} \forall_{k \geq 0}\ (x^j)^k = x^{jk}]$ [8-118]

158. $\forall_{x \neq 0} \forall_{y \neq 0} \forall_k\ (xy)^k = x^k y^k$ $[\forall_x \forall_y \forall_{k \geq 0}\ (xy)^k = x^k y^k]$ [8-119]

159. $\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}$ $[\forall_x \forall_{y \neq 0} \forall_{k \geq 0} \left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}]$ [8-119]

160. $\forall_{x \neq 0} \forall_{y \neq 0} \forall_k \left(\frac{x}{y}\right)^{-k} = \left(\frac{y}{x}\right)^k$ [8-119]

161. $\forall_{x>0} \forall_j \forall_k \, [x^j = x^k \iff (x = 1 \text{ or } j = k)]$ [page 8-122]

162. $\forall_{x \geq -1} \forall_{k \geq 0} \, (1 + x)^k \geq 1 + kx$ [8-125]

163. $\forall_{x>1} \forall_y \forall_n \, [n \geq \frac{y}{x-1} \Rightarrow x^n > y]$ [8-125]

164. $\forall_{x \neq 0} \, [\frac{1}{x} > 1 \iff 0 < x < 1]$ [8-125]

165. $\forall_{x \neq 1} \forall_{y>0} \forall_n \left[\left(0 < x < 1 \text{ and } n \geq \frac{1}{y(1-x)}\right) \Longrightarrow x^n < y\right]$ [8-126]

* * *

A sequence a is a geometric progression with common ratio r if and only if $a_1 \neq 0$ and $\forall_n \, a_{n+1} = a_n r$. [8-129 and 8-130]

* * *

166. $\forall_{x>0} \forall_{y>0} \, [x \neq y \Rightarrow \frac{x+y}{2} > \sqrt{xy}]$ [8-131]

167. If a is a GP with common ratio r, and, for each n, s_n is the sum of its first n terms, then

a. $\forall_n \, a_n = a_1 r^{n-1}$,

b. for $r \neq 0$, $\forall_n \, \frac{a_{n+1}}{a_n} = r$,

c. for $r \neq 1$, $\forall_n \, s_n = \frac{a_1(1 - r^n)}{1 - r}$,

d. for $r \neq 1$, $\forall_n \, s_n = \frac{a_1 - a_n r}{1 - r}$.

[8-130 and 8-133]

168 a. For any GP, a, with common ratio r such that $|r| < 1$,

$$\sum_{p=1}^{\infty} a_p = \frac{a_1}{1 - r}.$$

b. For each x such that $|x| < 1$, $\sum_{p=1}^{\infty} x^{p-1} = \frac{1}{1 - x}$.

[pages 8-144 and 8-146]

* * *

$$\forall_{x \geq 0}\ |x| = x \text{ and } \forall_{x \leq 0}\ |x| = -x$$ [8-145]

* * *

169 a. $\forall_x \forall_y\ |x| \cdot |y| = |xy|$

b. $\forall_x \forall_y\ [|x| < y \iff -y < x < y]$

c. $\forall_x \forall_y\ |x| - |y| \leq |x + y| \leq |x| + |y|$

[8-145]

170. $\forall_{k \geq 0} \forall_x \forall_y\ x^k - y^k = (x - y) \sum_{p=1}^{k} x^{k-p} y^{p-1}$ [8-160]

* * *

$$\begin{cases} \forall_{j \geq 0}\ C(j, 0) = 1 \\ \forall_{j \geq 0} \forall_{k \geq 0}\ C(j, k+1) = C(j, k) \cdot \frac{j - k}{k + 1} \end{cases}$$ [8-168]

* * *

$$\begin{cases} 0! = 1 \\ \forall_{k \geq 0}\ (k + 1)! = k! \cdot (k + 1) \end{cases}$$ [8-169]

* * *

171. $\forall_{j \geq 0} \forall_{k \geq 0}\ C(j, k) = \dfrac{\prod_{i=0}^{k-1} (j - 1)}{k!}$ [page 8-169]

172. $\forall_{j \geq 0} \forall_{k \geq 0}\ C(j+k, k) = \dfrac{(j+k)!}{j!\,k!}$ [8-171]

173. $\forall_m \forall_n\ C(m, n) = C(m - 1, n) + C(m - 1, n - 1)$ [8-172]

* * *

(C_1) Two sets have the same number of members if and only if the members of one set can be matched in a one-to-one way with those of the other. [8-173]

(C_2) If no two of a family of sets have a common member then the number of members in the union of the sets is the sum of the numbers of members in the individual sets. [8-174]

(C_3) If a first event can occur in any of m ways, and, after it has occurred, a second event can occur in any of n ways, then the number of ways in which the two events can occur successively is mn. [8-177]

(C_4) If a set A is the union of n subsets, $A_1, A_2, \ldots, A_n$ and if N_1 is the sum of the numbers of members of the subsets, N_2 is the sum of the numbers of members of the intersections of the subsets two at a time, N_3 is the sum of the numbers of members of the intersections of the subsets three at a time, etc., then [8-193]

$$N(A) = \sum_{i=1}^{n} (-1)^{i-1} N_i .$$

* * *

174. $\forall_{j \geq 0} \forall_{k \geq 0}\ P(j, k) = \prod_{i=0}^{k-1} (j - i)$ [page 8-180]

175. The number of permutations of p things, of which p_1 are of a first kind, p_2 of a second kind, ..., p_n are of an nth kind, and the remainder are of different kinds, is [8-183]

$$\frac{p!}{\prod_{q=1}^{n} p_q!}.$$

176. $\forall_{j \geq 0}\ C_j = 2^j$ [8-185]

177. The number of odd-membered subsets of a nonempty set is the same as the number of its even-membered subsets. [8-185]

178. $\forall_x \forall_y \forall_{j \geq 0} (x + y)^j = \sum_{k=0}^{j} C(j, k) x^{j-k} y^k$ [8-197]

179 a. $\forall_m \forall_n \sum_{p=1}^{n} C(p - 1, m - 1) = C(n, m)$ [8-205]

b. $\forall_{k \geq 0} \forall_n \sum_{p=1}^{n} C(k + p - 1, k) = C(n + k, k + 1)$

180. For any sequence a whose mth difference-sequence is a constant, [page 8-206]

a. $\forall_n \; a_n = a_1 + \sum_{k=1}^{m} C(n-1,\; k)(\Delta^k a)_1$

and

b. $\forall_n \; \sum_{p=1}^{n} a_p = na_1 + \sum_{k=1}^{m} C(n,\; k+1)(\Delta^k a)_1$.

181. Each composite number n has a prime divisor p such that $p^2 \leq n$. [8-207]

182. For any sequence n of positive integers, and any prime number p, [8-214]

$$\forall_m \left[p \,\Big|\, \prod_{i=1}^{m} n_i \Longrightarrow \exists_{q \leq m}\; p \,|\, n_q \right].$$

183. Each positive integer other than 1 has a unique prime factorization. [8-213]

TABLE OF TRIGONOMETRIC RATIOS

Angle	sin	cos	tan	Angle	sin	cos	tan
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

TABLE OF SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$
1	1	1.000	3.162	51	2601	7.141	22.583
2	4	1.414	4.472	52	2704	7.211	22.804
3	9	1.732	5.477	53	2809	7.280	23.022
4	16	2.000	6.325	54	2916	7.348	23.238
5	25	2.236	7.071	55	3025	7.416	23.452
6	36	2.449	7.746	56	3136	7.483	23.664
7	49	2.646	8.367	57	3249	7.550	23.875
8	64	2.828	8.944	58	3364	7.616	24.083
9	81	3.000	9.487	59	3481	7.681	24.290
10	100	3.162	10.000	60	3600	7.746	24.495
11	121	3.317	10.488	61	3721	7.810	24.698
12	144	3.464	10.954	62	3844	7.874	24.900
13	169	3.606	11.402	63	3969	7.937	25.100
14	196	3.742	11.832	64	4096	8.000	25.298
15	225	3.873	12.247	65	4225	8.062	25.495
16	256	4.000	12.649	66	4356	8.124	25.690
17	289	4.123	13.038	67	4489	8.185	25.884
18	324	4.243	13.416	68	4624	8.246	26.077
19	361	4.359	13.784	69	4761	8.307	26.268
20	400	4.472	14.142	70	4900	8.367	26.458
21	441	4.583	14.491	71	5041	8.426	26.646
22	484	4.690	14.832	72	5184	8.485	26.833
23	529	4.796	15.166	73	5329	8.544	27.019
24	576	4.899	15.492	74	5476	8.602	27.203
25	625	5.000	15.811	75	5625	8.660	27.386
26	676	5.099	16.125	76	5776	8.718	27.568
27	729	5.196	16.432	77	5929	8.775	27.749
28	784	5.292	16.733	78	6084	8.832	27.928
29	841	5.385	17.029	79	6241	8.888	28.107
30	900	5.477	17.321	80	6400	8.944	28.284
31	961	5.568	17.607	81	6561	9.000	28.460
32	1024	5.657	17.889	82	6724	9.055	28.636
33	1089	5.745	18.166	83	6889	9.110	28.810
34	1156	5.831	18.439	84	7056	9.165	28.983
35	1225	5.916	18.708	85	7225	9.220	29.155
36	1296	6.000	18.974	86	7396	9.274	29.326
37	1369	6.083	19.235	87	7569	9.327	29.496
38	1444	6.164	19.494	88	7744	9.381	29.665
39	1521	6.245	19.748	89	7921	9.434	29.833
40	1600	6.325	20.000	90	8100	9.487	30.000
41	1681	6.403	20.248	91	8281	9.539	30.166
42	1764	6.481	20.494	92	8464	9.592	30.332
43	1849	6.557	20.736	93	8649	9.644	30.496
44	1936	6.633	20.976	94	8836	9.695	30.659
45	2025	6.708	21.213	95	9025	9.747	30.822
46	2116	6.782	21.448	96	9216	9.798	30.984
47	2209	6.856	21.679	97	9409	9.849	31.145
48	2304	6.928	21.909	98	9604	9.899	31.305
49	2401	7.000	22.136	99	9801	9.950	31.464
50	2500	7.071	22.361	100	10000	10.000	31.623

TABLE OF CONSECUTIVE PRIMES

2	179	419	661	947	1229	1523	1823	2131	2437
3	181	421	673	953	1231	1531	1831	2137	2441
5	191	431	677	967	1237	1543	1847	2141	2447
7	193	433	683	971	1249	1549	1861	2143	2459
11	197	439	691	977	1259	1553	1867	2153	2467
13	199	443	701	983	1277	1559	1871	2161	2473
17	211	449	709	991	1279	1567	1873	2179	2477
19	223	457	719	997	1283	1571	1877	2203	2503
23	227	461	727	1009	1289	1579	1879	2207	2521
29	229	463	733	1013	1291	1583	1889	2213	2531
31	233	467	739	1019	1297	1597	1901	2221	2539
37	239	479	743	1021	1301	1601	1907	2237	2543
41	241	487	751	1031	1303	1607	1913	2239	2549
43	251	491	757	1033	1307	1609	1931	2243	2551
47	257	499	761	1039	1319	1613	1933	2251	2557
53	263	503	769	1049	1321	1619	1949	2267	2579
59	269	509	773	1051	1327	1621	1951	2269	2591
61	271	521	787	1061	1361	1627	1973	2273	2593
67	277	523	797	1063	1367	1637	1979	2281	2609
71	281	541	809	1069	1373	1657	1987	2287	2617
73	283	547	811	1087	1381	1663	1993	2293	2621
79	293	557	821	1091	1399	1667	1997	2297	2633
83	307	563	823	1093	1409	1669	1999	2309	2647
89	311	569	827	1097	1423	1693	2003	2311	2657
97	313	571	829	1103	1427	1697	2011	2333	2659
101	317	577	839	1109	1429	1699	2017	2339	2663
103	331	587	853	1117	1433	1709	2027	2341	2671
107	337	593	857	1123	1439	1721	2029	2347	2677
109	347	599	859	1129	1447	1723	2039	2351	2683
113	349	601	863	1151	1451	1733	2053	2357	2687
127	353	607	877	1153	1453	1741	2063	2371	2689
131	359	613	881	1163	1459	1747	2069	2377	2693
137	367	617	883	1171	1471	1753	2081	2381	2699
139	373	619	887	1181	1481	1759	2083	2383	2707
149	379	631	907	1187	1483	1777	2087	2389	2711
151	383	641	911	1193	1487	1783	2089	2393	2713
157	389	643	919	1201	1489	1787	2099	2399	2719
163	397	647	929	1213	1493	1789	2111	2411	2729
167	401	653	937	1217	1499	1801	2113	2417	2731
173	409	659	941	1223	1511	1811	2129	2423	2741